Wandering domains and Singularities

Núria Fagella

Facultat de Matemàtiques i Informàtica
Universitat de Barcelona
and
Barcelona Graduate School of Mathematics

Workshop on Complex Dynamics 2017
Deember 11-15, 2017

UNIVERSITAT DE BARCELONA

BGSMath
BARCELONA GRADUATE SCHOOL OF MATHEMATICS

Transcendental dynamics

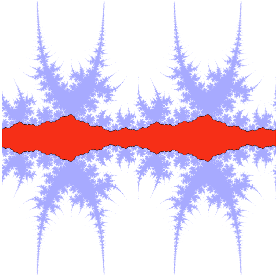
- If $f : \mathbb{C} \to \mathbb{C}$ (or to $\widehat{\mathbb{C}}$) has an essential singularity at infinity we say that f is transcendental.

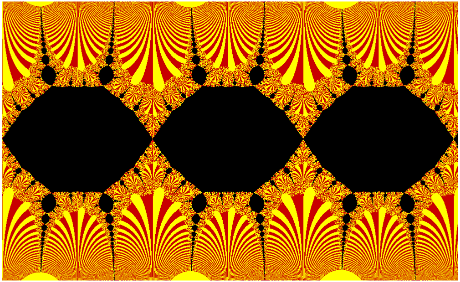
Transcendental dynamics

- If $f : \mathbb{C} \to \mathbb{C}$ (or to $\widehat{\mathbb{C}}$) has an essential singularity at infinity we say that f is transcendental.
- Transcendental maps may have Fatou components that are not basins of attraction nor rotation domains:
  - U is a Baker domain of period 1 if $f^n \mid_U \to \infty$ loc. unif.

Transcendental dynamics

- If $f : \mathbb{C} \to \mathbb{C}$ (or to $\widehat{\mathbb{C}}$) has an essential singularity at infinity we say that f is transcendental.
- Transcendental maps may have Fatou components that are not basins of attraction nor rotation domains:
  - U is a Baker domain of period 1 if $f^n\,|_U \to \infty$ loc. unif.
  - U is a wandering domain if $f^n(U) \cap f^m(U) = \emptyset$ for all $n \neq m$.





$z + a + b\sin(z)$ [Figures: Christian Henriksen] $z + 2\pi + \sin(z)$

Transcendental dynamics

- The set $S(f)$ of singularities of f^{-1} consists of critical values and asymtpotic values (and the closure of such).

Transcendental dynamics

- The set $S(f)$ of singularities of f^{-1} consists of critical values and asymtpotic values (and the closure of such).
- A critical value is the image of a critical point, i.e. $v = f(c)$ with $f'(c) = 0$.

Transcendental dynamics

- The set $S(f)$ of singularities of f^{-1} consists of critical values and asymtpotic values (and the closure of such).
- A critical value is the image of a critical point, i.e. $v = f(c)$ with $f'(c) = 0$.
- A point $a \in \widehat{\mathbb{C}}$ is an asymptotic value if there exists a curve $\gamma(t) \to \infty$ such that $f(\gamma(t)) \to a$. (Morally, a has infinitely many preimages collapsed at infinity).

  Example: $z = 0$ for $f(z) = \lambda e^z$.

Transcendental dynamics

- The set $S(f)$ of singularities of f^{-1} consists of critical values and asymtpotic values (and the closure of such).
- A critical value is the image of a critical point, i.e. $v = f(c)$ with $f'(c) = 0$.
- A point $a \in \widehat{\mathbb{C}}$ is an asymptotic value if there exists a curve $\gamma(t) \to \infty$ such that $f(\gamma(t)) \to a$. (Morally, a has infinitely many preimages collapsed at infinity).

  Example: $z = 0$ for $f(z) = \lambda e^z$.
- $f : \mathbb{C} \setminus f^{-1}(S(f)) \longrightarrow \mathbb{C} \setminus S(f)$ is a covering map of infinite degree.

Transcendental dynamics

- The set $S(f)$ of singularities of f^{-1} consists of critical values and asymtpotic values (and the closure of such).
- A critical value is the image of a critical point, i.e. $v = f(c)$ with $f'(c) = 0$.
- A point $a \in \widehat{\mathbb{C}}$ is an asymptotic value if there exists a curve $\gamma(t) \to \infty$ such that $f(\gamma(t)) \to a$. (Morally, a has infinitely many preimages collapsed at infinity).

  Example: $z = 0$ for $f(z) = \lambda e^z$.
- $f : \mathbb{C} \setminus f^{-1}(S(f)) \longrightarrow \mathbb{C} \setminus S(f)$ is a covering map of infinite degree.
- Define the postsingular set of f as

$$P(f) = \overline{\cup_{s \in S} \cup_{n \geq 0} f^n(s)}.$$

Singular values

Singular values play a very special role.

Singular values

Singular values play a very special role.

- Basins of attraction of attracting or parabolic cycles contain at least one singular value

$$U \cap P(f) \neq \emptyset$$

Singular values

Singular values play a very special role.

- Basins of attraction of attracting or parabolic cycles contain at least one singular value

$$U \cap P(f) \neq \emptyset$$

- Boundaries of Siegel disks belong to $P(f)$.

$$U \cap P(f) = \emptyset \text{ but } \partial U \subset P(f)$$

So iterating the singular values, S(f), one may find the periodic stable components".

Singular values

Singular values play a very special role.

- Basins of attraction of attracting or parabolic cycles contain at least one singular value

$$U \cap P(f) \neq \emptyset$$

- Boundaries of Siegel disks belong to $P(f)$.

$$U \cap P(f) = \emptyset \ \text{ but } \ \partial U \subset P(f)$$

So iterating the singular values, S(f), one may find the periodic stable components".

What about Baker and wandering domains?

Baker domains

The best result for Baker domains is the following.

Theorem (Bergweiler'95, Mihaljevic-rempe'13, Baranski-F-Jarque-Karpinska'17)

f transcendental meromorphic, U invariant Baker domain, $U \cap S(f) = \emptyset$. Then $\exists p_n \in P(f)$ st

1. $|p_n| \to \infty$
2. $\left|\frac{p_{n+1}}{p_n}\right| \to 1$
3. $\frac{\text{dist}(p_n, U)}{|p_n|} \to 0$

The theorem is sharp: there exists an (ETF) example for which $\text{dist}(p_n, U) > c > 0$.

Special classes

Some classes of maps are singled out depending on their singular values.

- The Speisser class or finite type maps:

$$\mathcal{S} = \{f \text{ ETF (or MTF) such that } S(f) \text{ is finite}\}$$

Example: $z \mapsto \lambda \sin(z)$

Special classes

Some classes of maps are singled out depending on their singular values.

- The Speisser class or finite type maps:

$$\mathcal{S} = \{f \text{ ETF (or MTF) such that } S(f) \text{ is finite}\}$$

Example: $z \mapsto \lambda \sin(z)$

Maps in S have NO WANDERING OR BAKER DOMAINS.

Special classes

- The Eremenko-Lyubich class

$$\mathcal{B} = \{f \text{ ETF (or MTF) such that } S(f) \text{ is bounded}\}$$

Example: $z \mapsto \lambda \frac{z}{\sin(z)}$.

Maps in class $\mathcal{B}$ have no Baker domains and NO ESCAPING WANDERING DOMAINS (Escaping set $\subset J(f)$).

Special classes

- The Eremenko-Lyubich class

$$\mathcal{B} = \{f \text{ ETF (or MTF) such that } S(f) \text{ is bounded}\}$$

Example: $z \mapsto \lambda \frac{z}{\sin(z)}$.

Maps in class $\mathcal{B}$ have no Baker domains and NO ESCAPING WANDERING DOMAINS (Escaping set $\subset J(f)$).

If U is a wandering domain, and $L(U)$ is the set of limit functions of f^n on U, then, all limit functions are constant and

$$U \text{ is} \begin{cases} \text{escaping} & \text{if } L(U) = \{\infty\} \\ \text{oscillating} & \text{if } \{\infty, a\} \subset L(U) \text{ for some } a \in \mathbb{C}. \\ \text{``bounded''} & \text{if } \infty \notin L(U). \end{cases}$$

Existence of wandering domains

Question

Can maps in class $\mathcal{B}$ have wandering domains at all?

Existence of wandering domains

Question

Can maps in class $\mathcal{B}$ have wandering domains at all?

Answer: yes.

Theorem (Bishop'15)

There exists an entire map $f \in \mathcal{B}$ such that f has an (oscillating) wandering domain.

Existence of wandering domains

Question

Can maps in class $\mathcal{B}$ have wandering domains at all?

Answer: yes.

Theorem (Bishop'15)

There exists an entire map $f \in \mathcal{B}$ such that f has an (oscillating) wandering domain.

- The proof is based on quasiconformal folding, a qc surgery construction.
- Incidentally, $U_n \cap P(f) \neq \emptyset$ for all n.

Existence of wandering domains

Question

Can maps in class $\mathcal{B}$ have wandering domains at all?

Answer: yes.

Theorem (Bishop'15)

There exists an entire map $f \in \mathcal{B}$ such that f has an (oscillating) wandering domain.

- The proof is based on quasiconformal folding, a qc surgery construction.
- Incidentally, $U_n \cap P(f) \neq \emptyset$ for all n.

Open question

Does there exist a map with a "bounded" wandering domain?

Examples of wandering domains

Examples of wandering domains are not abundant. Usual methods are:

- Lifiting of maps of $\mathbb{C}^*$ [Herman'89, Henriksen-F'09]. The relation with the singularities is limited to the finite type possibilities.

Examples of wandering domains

Examples of wandering domains are not abundant. Usual methods are:

- Lifiting of maps of $\mathbb{C}^*$ [Herman'89, Henriksen-F'09]. The relation with the singularities is limited to the finite type possibilities.
- Infinite products and clever modifications of known functions [Bergweiler'95, Rippon-Stallard'08'09...]

Examples of wandering domains

Examples of wandering domains are not abundant. Usual methods are:

- Lifiting of maps of $\mathbb{C}^*$ [Herman'89, Henriksen-F'09]. The relation with the singularities is limited to the finite type possibilities.
- Infinite products and clever modifications of known functions [Bergweiler'95, Rippon-Stallard'08'09...]
- Approximation theory [Eremenko-Lyubich'87]. No control on the singular values of the global map.

Examples of wandering domains

Examples of wandering domains are not abundant. Usual methods are:

- Lifiting of maps of $\mathbb{C}^*$ [Herman'89, Henriksen-F'09]. The relation with the singularities is limited to the finite type possibilities.
- Infinite products and clever modifications of known functions [Bergweiler'95, Rippon-Stallard'08'09...]
- Approximation theory [Eremenko-Lyubich'87]. No control on the singular values of the global map.
- Quasiconformal surgery [Kisaka-Shishikura'05, Bishop'15].

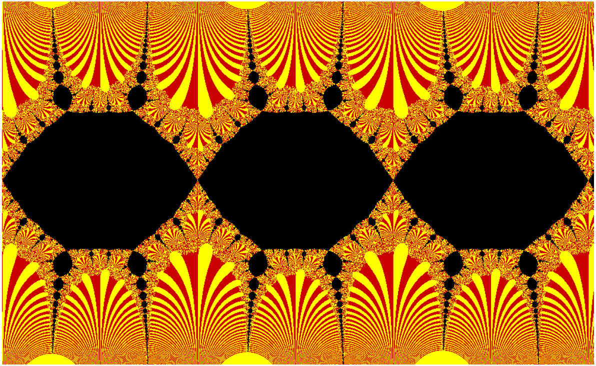
Wandering domains and singularities: Motivating examples

The relation of a wandering domain with the postcritical set is not so clear.

Wandering domains and singularities: Motivating examples

The relation of a wandering domain with the postcritical set is not so clear.

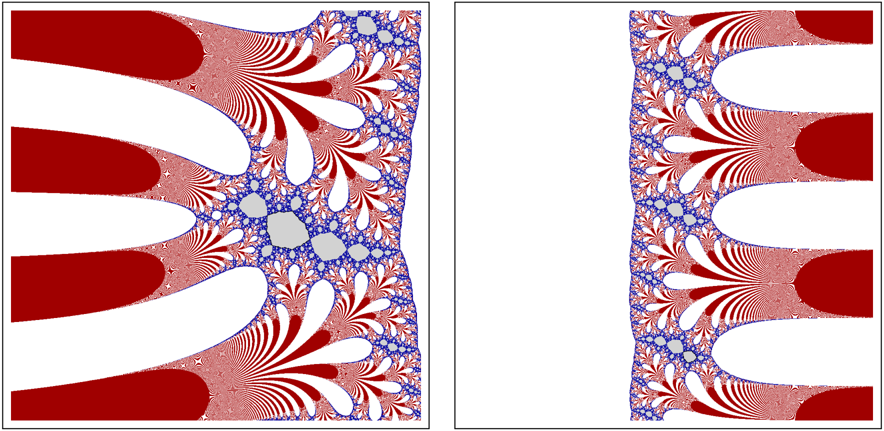
Example 1 (escaping):



$z \mapsto z + 2\pi + \sin(z)$ One critical point in each WD.

Wandering domains and singularities: Examples

Example 2 (escaping and Univalent, $\partial U \subset \overline{P(f)}$):



Left: Siegel disk of $g(w) = \frac{e^{2-\lambda}}{2-\lambda} w^2 e^{-w}$ with $\lambda = e^{2\pi i(1-\sqrt{5})/2)}$, around $w = 2 - \lambda$. Right: Lift to a wandering domain U.

Wandering domains and singularities: Examples

Example 3 [Kisaka-Shishilkura'05, Bergweiler-Rippon-Stallard'13]. Wandering orbit of annuli such that

- $\mathcal{U} \cap P(f) = \emptyset$
- $P(f) \subset F(f)$.

Wandering domains and singularities: Examples

Example 3 [Kisaka-Shishilkura'05, Bergweiler-Rippon-Stallard'13].
Wandering orbit of annuli such that

- $\mathcal{U} \cap P(f) = \emptyset$
- $P(f) \subset F(f)$.

Example 4 [Bishop'15]
The oscillating domain of Bishop in class $\mathcal{B}$ contains critical points of arbitrary high multiplicity, responsible for the high contraction necessary.

Wandering domains and singularities: Examples

Example 3 [Kisaka-Shishilkura'05, Bergweiler-Rippon-Stallard'13].
Wandering orbit of annuli such that

- $\mathcal{U} \cap P(f) = \emptyset$
- $P(f) \subset F(f)$.

Example 4 [Bishop'15]
The oscillating domain of Bishop in class $\mathcal{B}$ contains critical points of arbitrary high multiplicity, responsible for the high contraction necessary.

Question

Does there exist an oscillating wandering domain in class $\mathcal{B}$ on which f^n is univalent for all $n > 0$? (In part. $P(f) \cap U_n = \emptyset$?)

Known results

Recall, for U a wandering domain, the set of limit functions

$$L(U) = \{a \in \widehat{\mathbb{C}} \mid f^{n_k}|_U \rightrightarrows a \text{ for some } n_k \to \infty\}.$$

Theorem (Bergweiler *et al*'93, Baker'02, Zheng'03)

Let f be a MTF with a wandering domain U. If $a \in L(U)$ then $a \in P(f)' \cap J(f)$.

Known results

Recall, for U a wandering domain, the set of limit functions

$$L(U) = \{a \in \widehat{\mathbb{C}} \mid f^{n_k}|_U \rightrightarrows a \text{ for some } n_k \to \infty\}.$$

Theorem (Bergweiler *et al*'93, Baker'02, Zheng'03)

Let f be a MTF with a wandering domain U. If $a \in L(U)$ then $a \in P(f)' \cap J(f)$.

Theorem (Mihaljevic-Rempe'13)

If $f \in \mathcal{B}$ and $f^n(S(f)) \rightrightarrows \infty$ uniformly (+ extra geometric assumption), then f has no wandering domains.

Wandering domains and singular orbits

Theorem B (Baranski-F-Jarque-Karpinska'17)

Let f be a MTF and U be a wandering domain of f. Let U_n be the Fatou component such that $f^n(U) \subset U_n$. Then for every $z \in U$ there exists a sequence $p_n \in P(f)$ such that

$$\frac{\operatorname{dist}(p_n, U_n)}{\operatorname{dist}(f^n(z), \partial U_n)} \to 0 \quad \text{as } n \to \infty.$$

In particular, if for some $d > 0$ we have $\operatorname{dist}(f^n(z), \partial U_n) < d$ for all n (for instance if the diameter of U_n is uniformly bounded), then $\operatorname{dist}(p_n, U_n) \to 0$ as n tends to ∞.

Wandering domains and singular orbits

Theorem B (Baranski-F-Jarque-Karpinska'17)

Let f be a MTF and U be a wandering domain of f. Let U_n be the Fatou component such that $f^n(U) \subset U_n$. Then for every $z \in U$ there exists a sequence $p_n \in P(f)$ such that

$$\frac{\operatorname{dist}(p_n, U_n)}{\operatorname{dist}(f^n(z), \partial U_n)} \to 0 \quad \text{as } n \to \infty.$$

In particular, if for some $d > 0$ we have $\operatorname{dist}(f^n(z), \partial U_n) < d$ for all n (for instance if the diameter of U_n is uniformly bounded), then $\operatorname{dist}(p_n, U_n) \to 0$ as n tends to ∞.

Proof: normal families argument, hyperbolic geometry.... Based on the improvement of a technical lemma from Bergweiler on Baker domains. Compare also [Mihaljevic-Rempe'13]. ▸ More details.

Application: Topologically hyperbolic functions

- A MTF is topologically hyperbolic if

$$\operatorname{dist}(P(f), J(f) \cap \mathbb{C}) > 0.$$

Application: Topologically hyperbolic functions

- A MTF is topologically hyperbolic if

$$\text{dist}(P(f), J(f) \cap \mathbb{C}) > 0.$$

- This condition can be regarded as a kind of weak hyperbolicity in the context of transcendental meromorphic functions since $|(f^n)'(z)| \to \infty$ for all $z \in J(f)$ [Stallard'90, Mayer-Urnbanski'07'10].

Application: Topologically hyperbolic functions

- A MTF is topologically hyperbolic if

$$\text{dist}(P(f), J(f) \cap \mathbb{C}) > 0.$$

- This condition can be regarded as a kind of weak hyperbolicity in the context of transcendental meromorphic functions since $|(f^n)'(z)| \to \infty$ for all $z \in J(f)$ [Stallard'90, Mayer-Urnbanski'07'10].
- Topologically hyperbolic maps do not possess parabolic cycles, rotation domains or wandering domains which do not tend to infinity

Application: Topologically hyperbolic functions

- A MTF is topologically hyperbolic if

$$\mathrm{dist}(P(f), J(f) \cap \mathbb{C}) > 0.$$

- This condition can be regarded as a kind of weak hyperbolicity in the context of transcendental meromorphic functions since $|(f^n)'(z)| \to \infty$ for all $z \in J(f)$ [Stallard'90, Mayer-Urnbanski'07'10].
- Topologically hyperbolic maps do not possess parabolic cycles, rotation domains or wandering domains which do not tend to infinity
- Examples include many Newton's methods of entire functions.

Application: Topologically hyperbolic functions

Corollary C

Let f be a MTF topologically hyperbolic. Let U be a wandering domain s.t. $U_n \cap P(f) = \emptyset$ for $n > 0$. Then for every compact set $K \subset U$ and every $r > 0$ there exists n_0 such that for every $z \in K$ and every $n \geq n_0$,

$$\mathbb{D}(f^n(z), r) \subset U_n.$$

In particular,

$$\operatorname{diam} U_n \to \infty \qquad \textit{and} \qquad \operatorname{dist}(f^n(z), \partial U_n) \to \infty$$

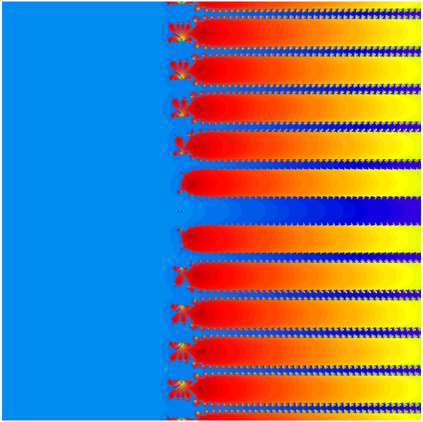
for every $z \in U$, as $n \to \infty$.

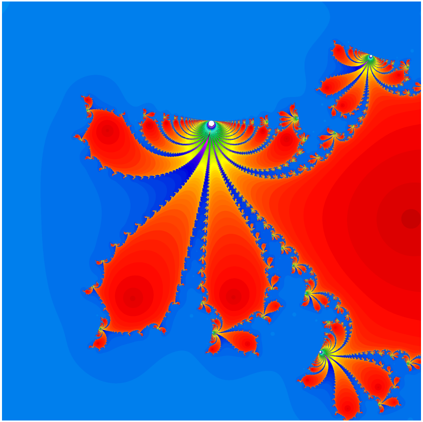
This can be applied to show that many functions, including Newton's method of $h(z) = ae^z + bz + c$ with $a, b, c \in \mathbb{R}$, have no wandering domains

[c.f. Bergweiler-Terglane,Kriete].

No wandering domains

Newton's method for $F(z) = z + e^z$.





Univalent WD in class $\mathcal{B}$

Theorem A (F-Lazebnik-Jarque'17)

There exists an ETF $f \in \mathcal{B}$ such that f has a wandering domain U on which $f^n|_U$ is univalent for all $n \geq 0$.

Univalent WD in class $\mathcal{B}$

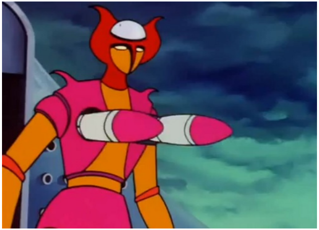
Theorem A (F-Lazebnik-Jarque'17)

There exists an ETF $f \in \mathcal{B}$ such that f has a wandering domain U on which $f^n|_U$ is univalent for all $n \geq 0$.

The proof is based on Bishop's quasiconformal folding construction.
We substitute the high degree maps $(z - z_n)^{d_n}$ on the disk components by $(z - z_n)^{d_n} + \delta_n(z - z_n)$, which are univalent near z_n and show that that the critical values can be kept outside (but very close to) the wandering component.

▸ More detail

Thank you for your attention!



Technical lemma

The technical lemma on the proof is the following.

Lemma

f TMF, U wandering domain, $U_n = f^n(U)$. Then,
$\forall K$ compact, $\varepsilon > 0$, $M \geq 1$,
there exists n_0 such that
for all $n > n_0$, $z \in K$, γ curve connecting $f^n(z)$ to $w \in \partial U$ with

$$\text{length}(\gamma) \leq \text{dist}(f^n(z), \partial U_n)$$

there exists

$$p \in \mathbb{D}(\gamma, \varepsilon\, \text{length}(\gamma)) \cap P(f).$$

Go back