A wandering domain in class $\mathcal{B}$ on which all iterates are univalent

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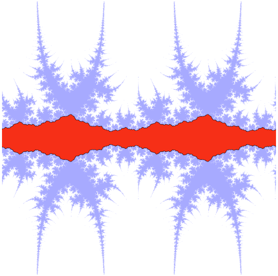
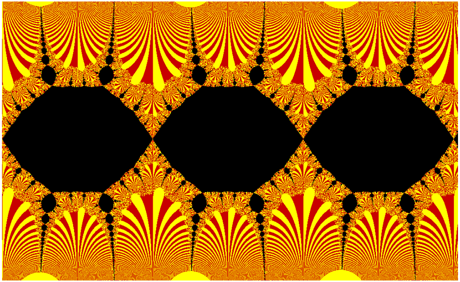
  - U is a Baker domain of period 1 if $f^n|_U \to \infty$ loc. unif.
  - U is a wandering domain if $f^n(U) \cap f^m(U) = \emptyset$ for all $n \neq m$.





$z + a + b\sin(z)$ [Figures: Christian Henriksen] $z + 2\pi + \sin(z)$

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- $f : \mathbb{C} \setminus f^{-1}(S(f)) \longrightarrow \mathbb{C} \setminus S(f)$ is a covering map of infinite degree.
- Define the postsingular set of f as

$$P(f) = \overline{\cup_{s \in S} \cup_{n \geq 0} f^n(s)}.$$

Special classes

- The Eremenko-Lyubich class

$$\mathcal{B} = \{f \text{ ETF (or MTF) such that } S(f) \text{ is bounded}\}$$

Example: $z \mapsto \lambda \frac{z}{\sin(z)}$.

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If U is a wandering domain, and $L(U)$ is the set of limit functions of f^n on U, then, all limit functions are constant and

$$U \text{ is } \begin{cases} \text{escaping} & \text{if } L(U) = \{\infty\} \\ \text{oscillating} & \text{if } \{\infty, a\} \subset L(U) \text{ for some } a \in \mathbb{C}. \\ \text{“bounded”} & \text{if } \infty \notin L(U). \end{cases}$$

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Question

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Theorem A (F-Lazebnik-Jarque'17)

There exists an ETF $f \in \mathcal{B}$ such that f has a wandering domain U on which $f^n|_U$ is univalent for all $n \geq 0$.

Constructing entire functions

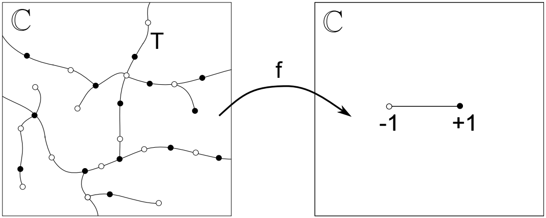
Bishop's qc-folding construction

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function with

- exactly two critical values, say -1 and $+1$
- no finite asymptotic values

Question: What does f look like ?

$T = f^{-1}([-1, +1])$ is an infinite bipartite tree.



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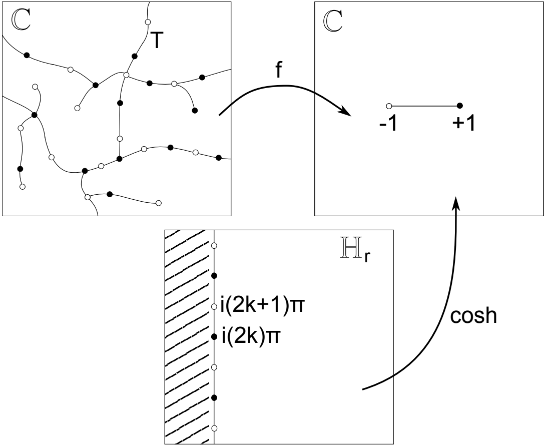

cosh : $\mathbb{H}_r \to \mathbb{C}\backslash[-1,+1]$ is a universal cover.

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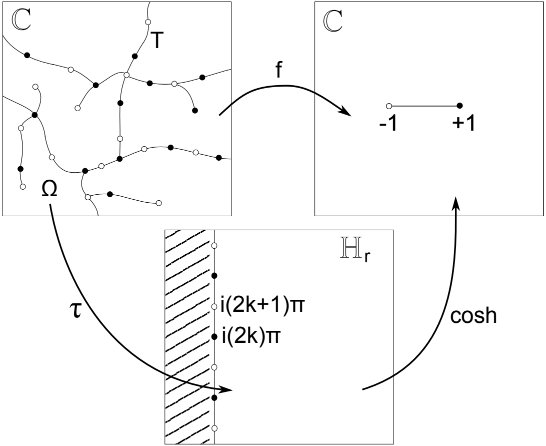

$\forall \Omega$ c.c. of $\mathbb{C} \setminus T$, $\tau_{|\Omega} = (\cosh^{-1} \circ f_{|\Omega}) : \Omega \to \mathbb{H}_r$ is conformal.

Conversely: How to construct f from (T, τ) ?

More precisely: Given

- an infinite bipartite tree $T \subset \mathbb{C}$ with "good enough" geometry
- a map τ such that $\tau_{|\Omega} : \Omega \to \mathbb{H}_r$ is conformal, $\forall \Omega$ c.c. of $\mathbb{C} \setminus T$

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Question: Does there exist an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f = \cosh \circ \tau$?

Main problem: In general, $\cosh \circ \tau$ is not continuous across T.

Strategy:

Step 1: Modify (T, τ) in a small neighborhood $T(r_0)$ of T.

More precisely, replace (T, τ) by (T', η) such that

- $T \subset T' \subset T(r_0)$
- $\eta = \tau$ off $T(r_0)$
- $\eta_{|\Omega'} : \Omega' \to \mathbb{H}_r$ is K-quasiconformal, $\forall \Omega'$ c.c. of $\mathbb{C} \setminus T'$
- $\cosh \circ \eta$ continuous across T' (quasiregular map)

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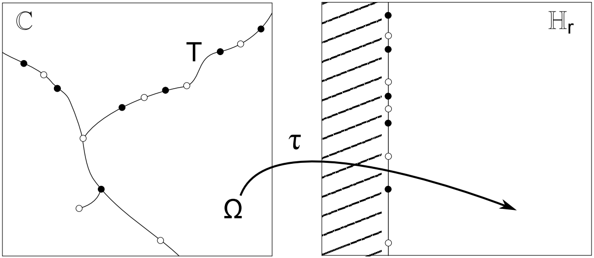
Step 2: Apply MRMT .

Obtain a qc map ϕ (the integrating map of $\mu_{\cosh \circ \eta}$) so that $f := \cosh \circ \eta \circ \phi^{-1}$ is entire. In particular

$$f \circ \phi = \cosh \circ \tau \text{ off } T(r_0)$$

Bishop's qc-folding construction



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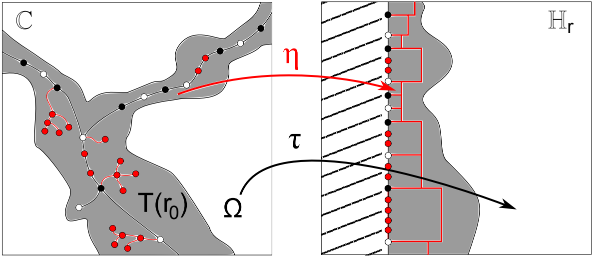

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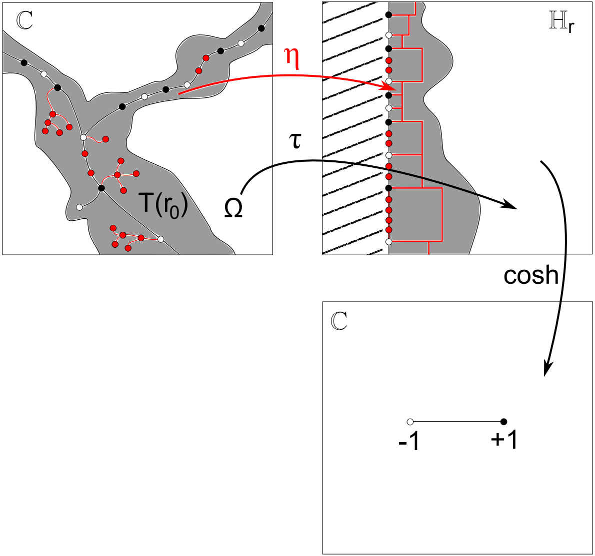

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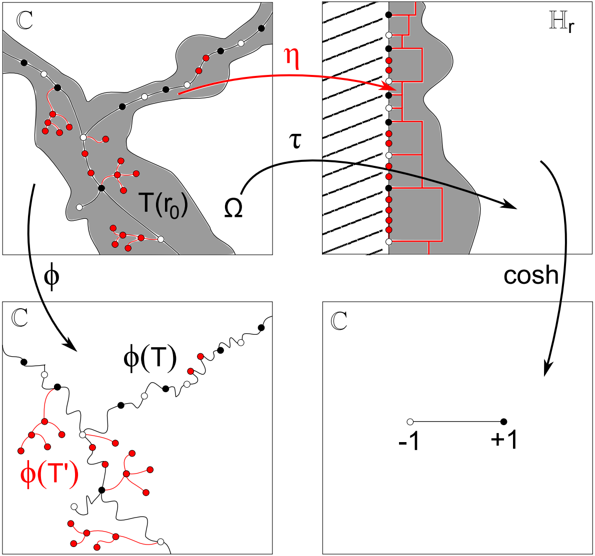

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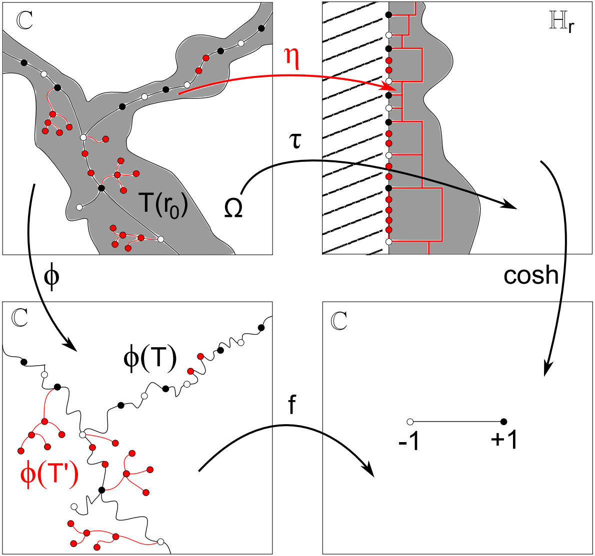
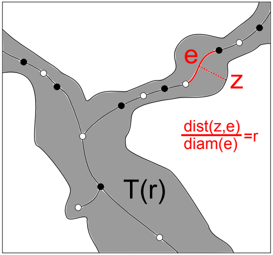


Bounded geometry

Definition: We say that T has bounded geometry if

- edges of T are $\mathcal{C}^2$ with uniform bounds
- angles between adjacent edges are uniformly bounded away from 0
- $\forall e, f$ adjacent edges, $\frac{1}{M} \leqslant \frac{\text{diam}(e)}{\text{diam}(f)} \leqslant M$
- $\forall e, f$ non-adjacent edges, $\frac{\text{diam}(e)}{\text{dist}(e,f)} \leqslant M$

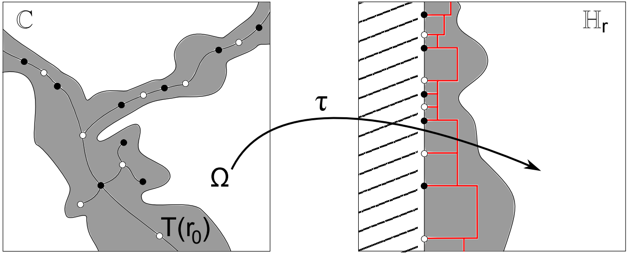


$$T(r) = \bigcup_{e \text{ edge of } T} \left\{ z \in \mathbb{C} \;/\; \text{dist}(z,e) < r\,\text{diam}(e) \right\}$$

Lemma

If T has bounded geometry, then $\exists r_0 > 0$ such that

$$\forall \Omega \text{ c.c. of } \mathbb{C}\setminus T,\ \forall \text{ square } Q \subset \mathbb{H}_r \text{ that has a } \tau_{|\Omega}\text{-edge as one side,}$$
$$Q \subset \tau_{|\Omega}\Big(T(r_0) \cap \Omega\Big)$$



Every edge has two τ-sizes!!!

Bishop's Theorem

Theorem (Bishop'12)

If (T, τ) satisfies the following conditions

1. *T has bounded geometry*
2. *every edge has τ-size $\geqslant \pi$*

then $\exists$ an entire function f and a quasiconformal map ϕ such that

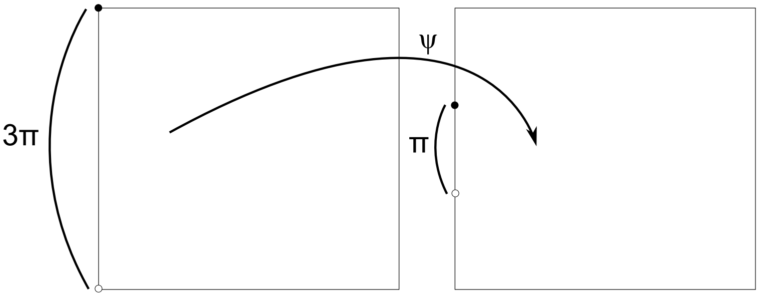
$$f \circ \phi = \cosh \circ \tau \text{ off } T(r_0)$$

Moreover

- *f has exactly two critical values, -1 and $+1$*
- *f has no finite asymptotic values*
- $\phi(T) \subset f^{-1}([-1, +1]) \quad (= \phi(T'))$
- *$\forall c$ critical point of f,* $\deg_{\text{loc}}(c, f) = \deg(c, \phi(T'))$

The main technical difficulty is to find a quasiconformal map ψ from a square to itself such that

$$\begin{cases} \psi \text{ maps the left side to an edge of length } \pi \\ \psi \text{ is the identity on the right side} \end{cases}$$



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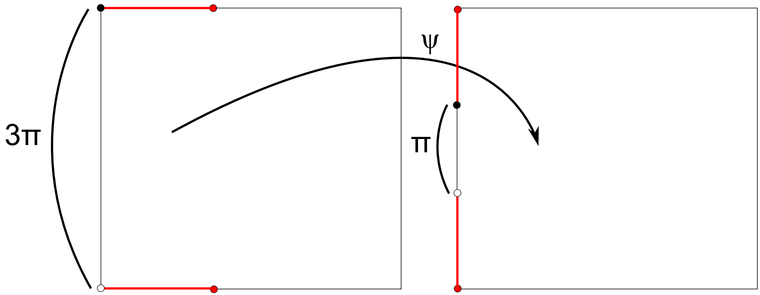

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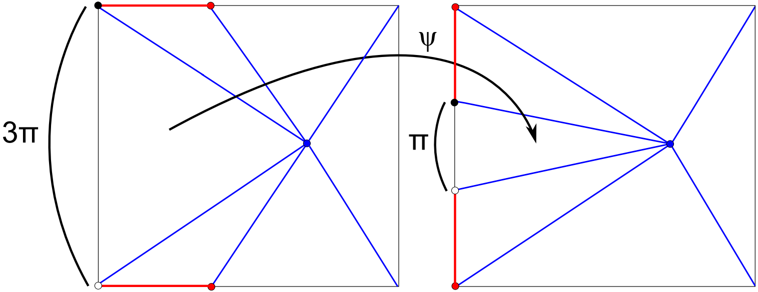

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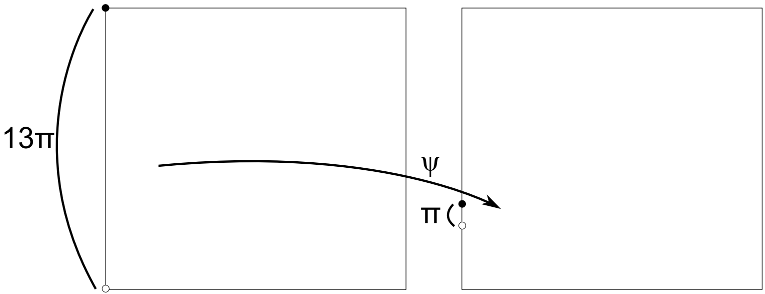

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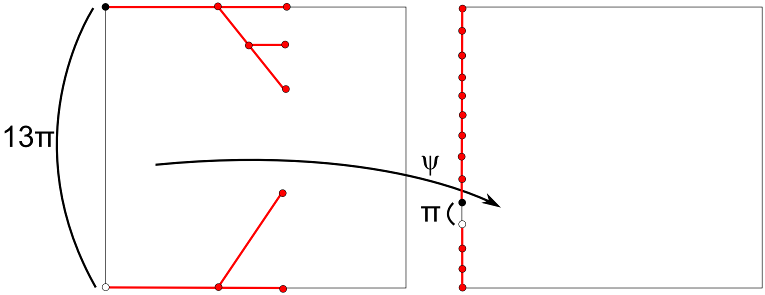

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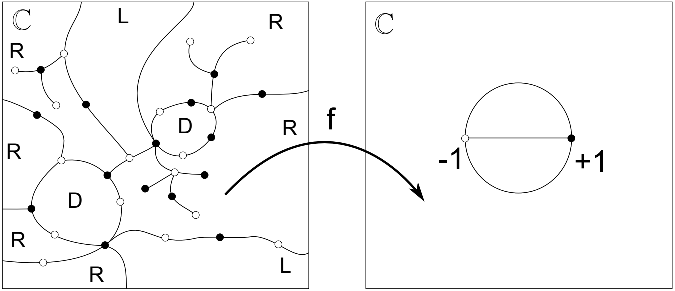
Generalization: We may also construct f with

- more critical values than only -1 and $+1$
- some finite asymptotic values
- arbitrary high degree critical points

Let T be an infinite bipartite graph.



The c.c. of $\mathbb{C}\setminus T$ are sorted into three different types:

R-components: $\tau_{|\Omega} : \Omega \to \mathbb{H}_r$ conformally
D-components: $\tau_{|\Omega} : \Omega \to \mathbb{D}$ conformally
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R	$\Omega \xrightarrow{\tau_{|\Omega}} \mathbb{H}_r \xrightarrow{\cosh}$
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R	Ω	$\xrightarrow{\tau_{\vert\Omega}}$	$\mathbb{H}_r$	$\xrightarrow{\cosh}$	$\mathbb{C} \setminus [-1, +1]$
D	$(\Omega, \star)$	$\xrightarrow{\tau_{\vert\Omega}}$	$(\mathbb{D}, 0)$		
L	Ω	$\xrightarrow{\tau_{\vert\Omega}}$	$\mathbb{H}_\ell$		

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D	$(\Omega,\star)$	$\xrightarrow{\tau_{\vert\Omega}}$	$(\mathbb{D},0)$	$\xrightarrow{z\mapsto z^{d_\Omega}}$	$(\mathbb{D},0)$	$\xrightarrow{\rho_\Omega}$	$(\mathbb{D},w_\Omega)$
L	(Ω,∞)	$\xrightarrow{\tau_{\vert\Omega}}$	$(\mathbb{H}_\ell,-\infty)$				

where $\rho_\Omega : \mathbb{D} \to \mathbb{D}$ is quasiconformal with $\rho_\Omega(z) = z$, $\forall z \in \partial\mathbb{D}$.

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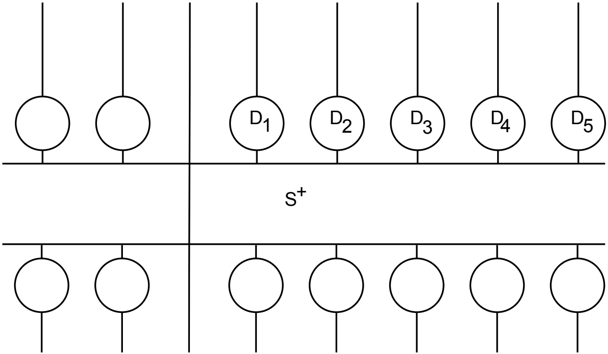
L	(Ω,∞)	$\xrightarrow{\tau_{\mid\Omega}}$	$(\mathbb{H}_\ell,-\infty)$	$\xrightarrow{\exp}$	$(\mathbb{D},0)$	$\xrightarrow{\rho_\Omega}$	$(\mathbb{D},v_\Omega)$

Constructiong the oscillating wandering domains in class $\mathcal{B}$

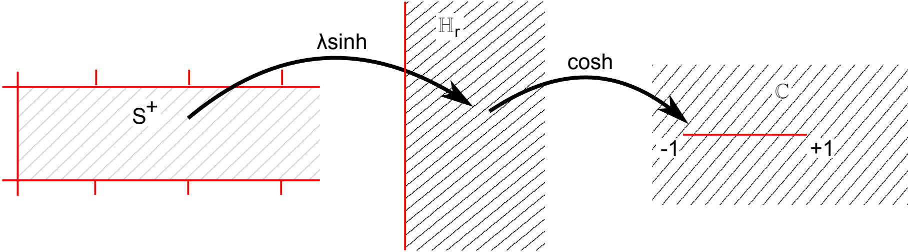
$$f = F \circ \phi^{-1} \quad \text{with} \quad \begin{cases} F : \mathbb{C} \to \mathbb{C} \text{ quasiregular (transcendental)} \\ \phi : \mathbb{C} \to \mathbb{C} \text{ quasiconformal so that } \phi^* \mu_0 = F^*(\mu_0) \end{cases}$$

F is constructed using an infinite graph.



Oscillating wandering domains in class $\mathcal{B}$

$$F: \quad S^+ \xrightarrow{\lambda \sinh} \mathbb{H}_r \xrightarrow{\cosh} \mathbb{C}\setminus[-1,+1]$$
$$z \longmapsto \cosh(\lambda \sinh(z))$$



$\boxed{\lambda > 0}$ is fixed so that $f^n\left(\frac{1}{2}\right) \xrightarrow[n\to+\infty]{} +\infty$ very fast.

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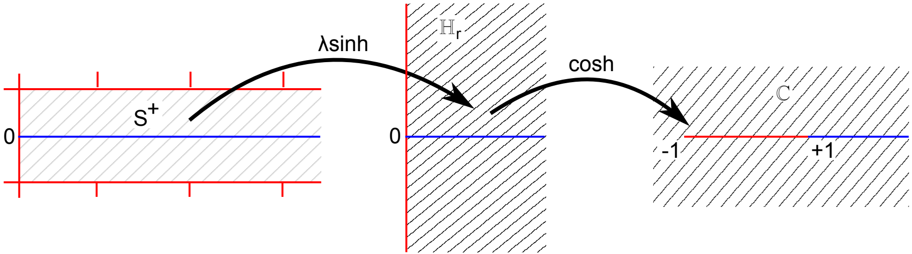

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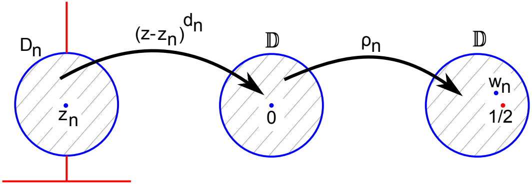
For every $n \geqslant 1$,

$$F: \quad (D_n, z_n) \xrightarrow{z \mapsto (z - z_n)^{d_n}} (\mathbb{D}, 0) \xrightarrow{\rho_n} (\mathbb{D}, w_n)$$
$$z \longmapsto \rho_n\left((z - z_n)^{d_n}\right)$$

with $\begin{cases} \rho_n : \mathbb{D} \to \mathbb{D} \text{ quasiconformal} \\ \rho_n(0) = w_n \end{cases}$

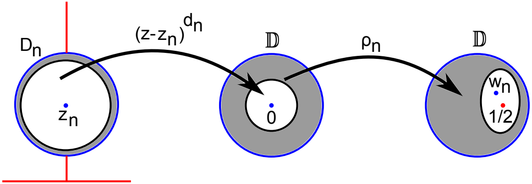


for some parameters $\boxed{d_n \xrightarrow[n \to +\infty]{} +\infty}$ and $\boxed{w_n \xrightarrow[n \to \infty]{} \frac{1}{2}}$.

For every $n \geqslant 1$,

$$\begin{array}{ccccc} F: & (D_n, z_n) & \xrightarrow{z \mapsto (z-z_n)^{d_n}} & (\mathbb{D}, 0) & \xrightarrow{\rho_n} & (\mathbb{D}, w_n) \\ & z & \longmapsto & & & \rho_n\left((z-z_n)^{d_n}\right) \end{array}$$

with $\begin{cases} \rho_n : \mathbb{D} \to \mathbb{D} \text{ quasiconformal} \\ \rho_n(0) = w_n \\ \mathrm{supp}(\mu_{\rho_n}) \subset \left\{\frac{1}{2} \leqslant |z| \leqslant 1\right\} \end{cases}$



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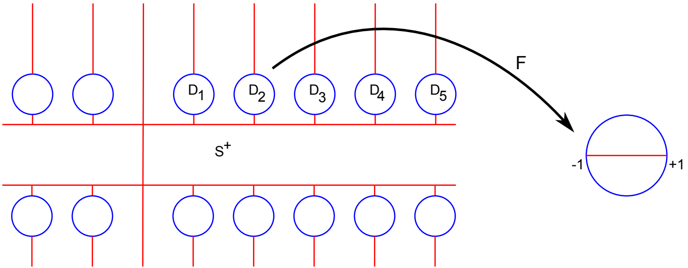
Using Bishop's construction F may be extended to a quasiregular map $F : \mathbb{C} \to \mathbb{C}$ such that:

- $\forall z \in \mathbb{C}$, $F(-z) = F(z)$ and $F(\bar{z}) = \overline{F(z)}$
- $\mathrm{Crit}(F) = \{-1, +1\} \cup \{w_n,\ n \geqslant 1\} \cup \left\{\frac{1}{2}\right\} \subset \overline{\mathbb{D}}$ with $w_n \xrightarrow[n \to +\infty]{} \frac{1}{2}$
- $\mathrm{Asym}(F) = \emptyset$



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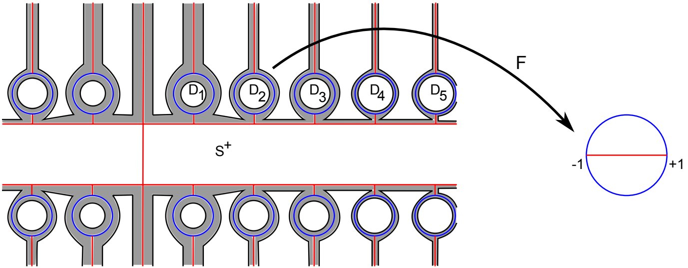

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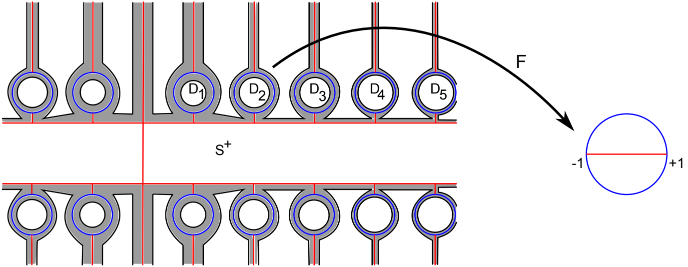
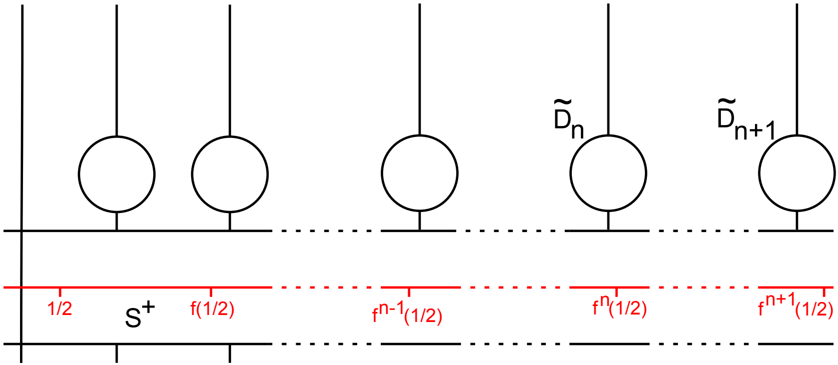
- $\mathrm{Asym}(F) = \emptyset$
- $\mathrm{supp}\,(F^*(\mu_0))$ is small enough in order that $\phi|_{S^+} \approx \mathrm{Id}_{S^+}$



Let $f = F \circ \phi$ with $\phi : \mathbb{C} \to \mathbb{C}$ quasiconformal so that $\mu_{\phi^{-1}} = F^*(\mu_0)$.

Oscillating wandering domains in class $\mathcal{B}$



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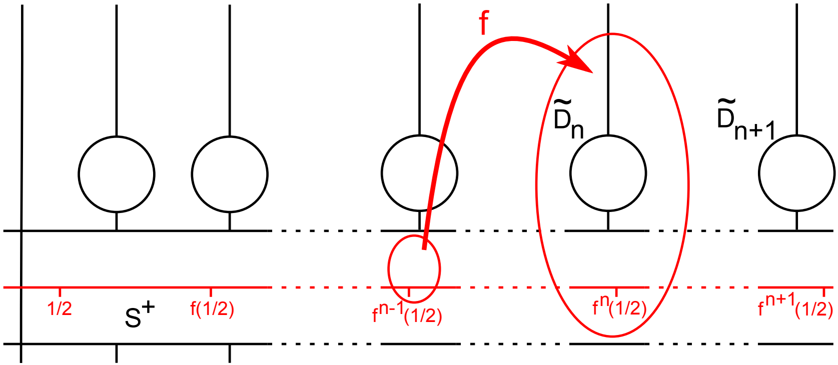

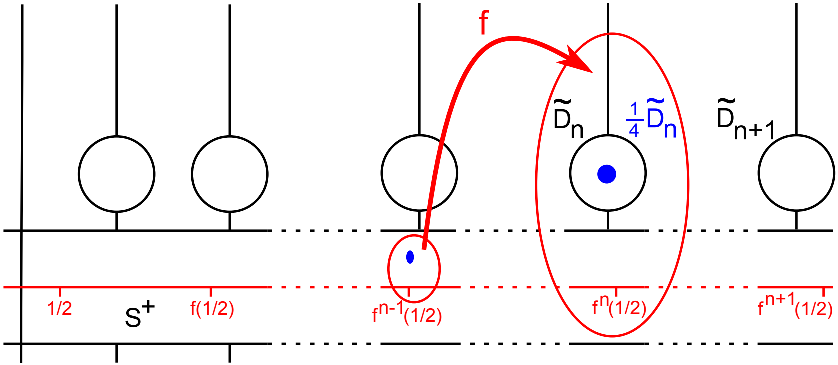
f
D̃n
¼D̃n
D̃n+1
1/2
S+
f(1/2)
fn-1(1/2)
fn(1/2)
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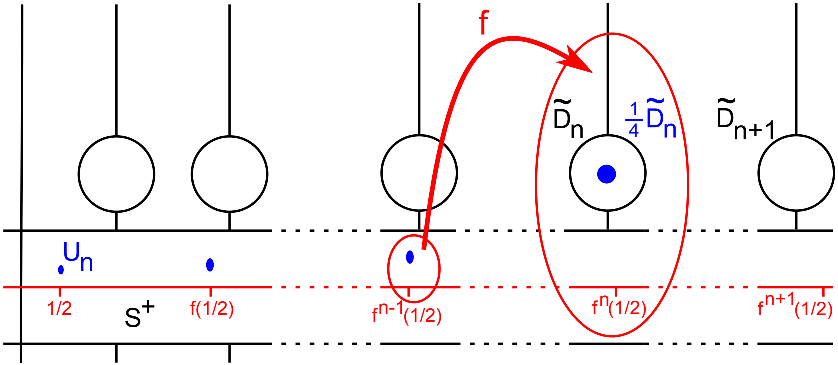

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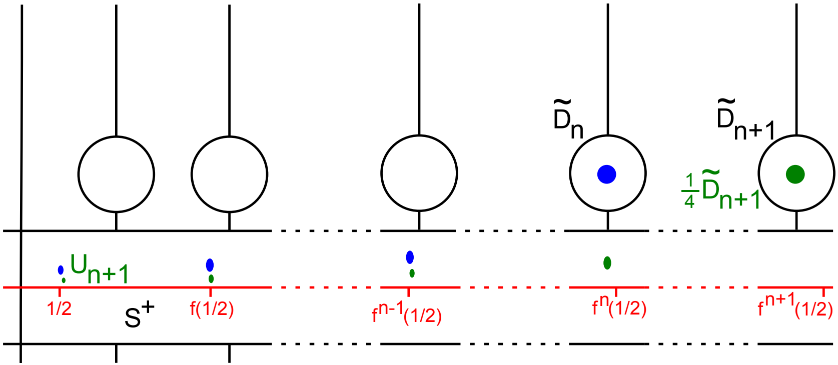

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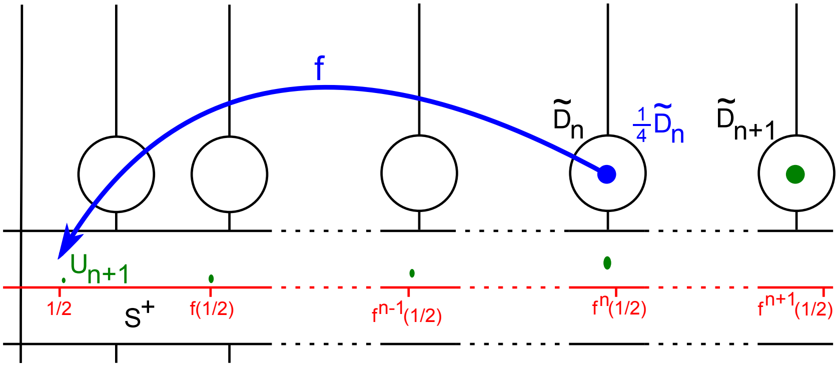

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$$\widetilde{w}_n \in f\left(\frac{1}{4}\widetilde{D}_n\right) \quad \text{and} \quad \mathrm{diam}\left(f\left(\frac{1}{4}\widetilde{D}_n\right)\right) \leqslant C'.\left(\frac{1}{4}\right)^{\widetilde{d}_n}$$

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- The parameters w_n, d_n are adjusted successively obtaining a new map f, everytime closer to the previous one, converging to a final map f.
- The correction map ϕ at every step is closer and closer to the identity, since the support of μ decreases exponentially.

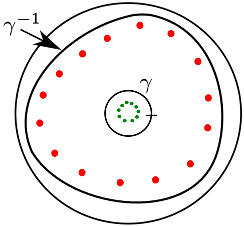
Modifying F to construct a univalent wandering domain

The new map on $D-$components

Consider the map

$$\begin{array}{ccc} \mathbb{D} & \longrightarrow & \mathbb{D} \\ z & \longmapsto & z^m + \delta z \end{array}$$

- $m-1$ critical points which tend to $\partial\mathbb{D}$ when $m \to \infty$
- $m-1$ critical values which tend to δ in modulus when $m \to \infty$.
- Let $\gamma = \{|z| = \frac{3}{2}\}$ and then $\mathrm{int}(\gamma^{-1})$ contains all critical points.



Define a quasiregular map ψ on $\mathbb{D}$

$$\psi_m = \begin{cases} z^m + \delta z, & \text{if } z \in \operatorname{int}\left(\gamma^{-1}\right) \\ z^m & \text{if } z \in \partial\mathbb{D}, \end{cases}$$

and linear interpolation in between $\operatorname{int}\left(\gamma^{-1}\right)$ and $\partial\mathbb{D}$.

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Claim 1

The dilatation of ψ_m is

- *uniformly bounded independently of m and $\delta << \frac{1}{2}$.*
- *supported on a region whose area tends to 0 as $m \to \infty$.*

▸ Idea of the proof

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- Compose with ρ_n sending 0 to w_n near $\frac{1}{2}$. The critical values of $\rho_n \circ \phi_n(z - z_n)$ are now centered around w_n.
- Bishop's theorem yields a quasiregular extension of g in $\mathbb{C}$ with dilatation independent of the parameters, and an entire map $f = g \circ \phi$, with ϕ qc.

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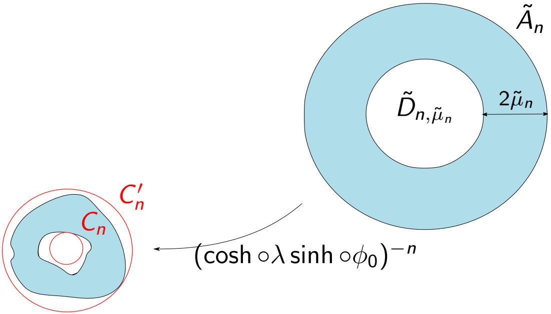
  - the process converges
  - the modifications do not change the dynamics we want to preserve.

We consider straight annuli around ∂D_n (the safety belt)

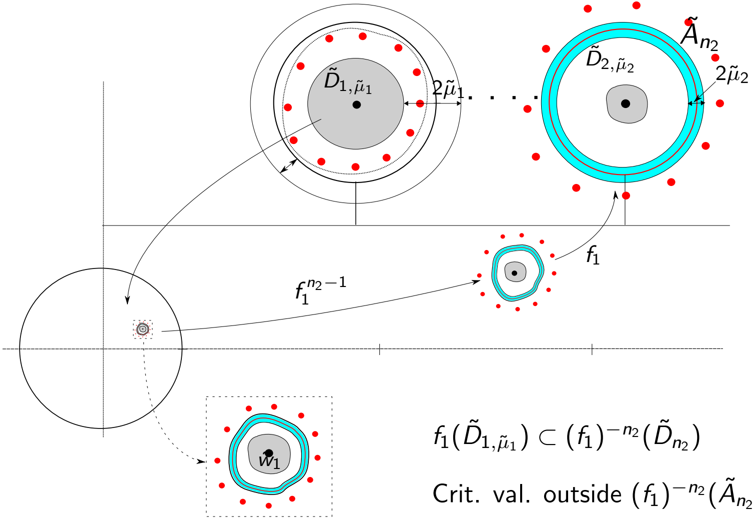
$$A_n = \{1 - \mu_n < |z - z_n| < 1 + \mu_n\}.$$



Claim 2

$$\frac{\mathrm{radius}(\mathrm{C_n'})}{\mathrm{radius}(\mathrm{C_n})} \underset{n\to\infty}{\longrightarrow} \frac{1+\tilde{\mu}_n}{1-\tilde{\mu}_n}$$

We adjust $\mu_2 << \mu_1$, $w_1 = f^{-n_2}(\tilde{z}_n)$ and δ_1 appropriately so that:



$f_1(\tilde{D}_{1,\tilde{\mu}_1}) \subset (f_1)^{-n_2}(\tilde{D}_{n_2})$

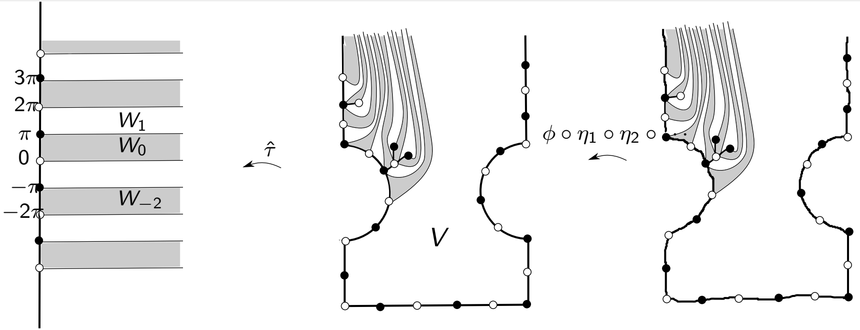
Crit. val. outside $(f_1)^{-n_2}(\tilde{A}_{n_2})$

Claim 3

There exists a wandering domain U containing $\tilde{D}_{1,\tilde{\mu}_1}$, and it satisfies

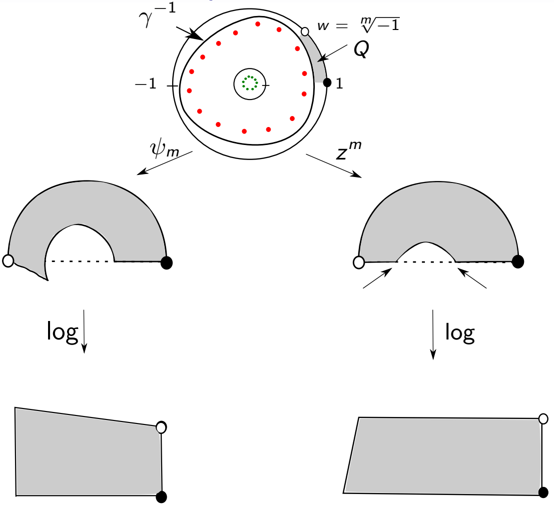
$$U_n \subset\subset \mathbb{D}.$$

In particular, its orbit does not contain singular values and therefore $f^n|_U$ is univalent for all n.



Thank you for your attention!

Uniformly constant dilatation



Estimate $\mathrm{mod}Q$ (right) and $\mathrm{mod}(\psi_m(Q))$ (left) and see that the quotient is uniformly bounded when $m \to \infty$ and $\delta \to 0$.

Go back