Dynamical & Parametric Zalcman Functions: Similarity between the Julia sets, the Mandelbrot set, & the tricorn

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— In Memory of Tan Lei —

(1963 - 2016)

1 Zalcman's Lemma

$$\begin{cases} D \subset \mathbb{C}, & \text{a domain} \\ \mathcal{F}: & \text{a family of meromorphic functions on } D \end{cases}$$

<u>Zalcman's Lemma</u> ('73)

in any nbd of z_0

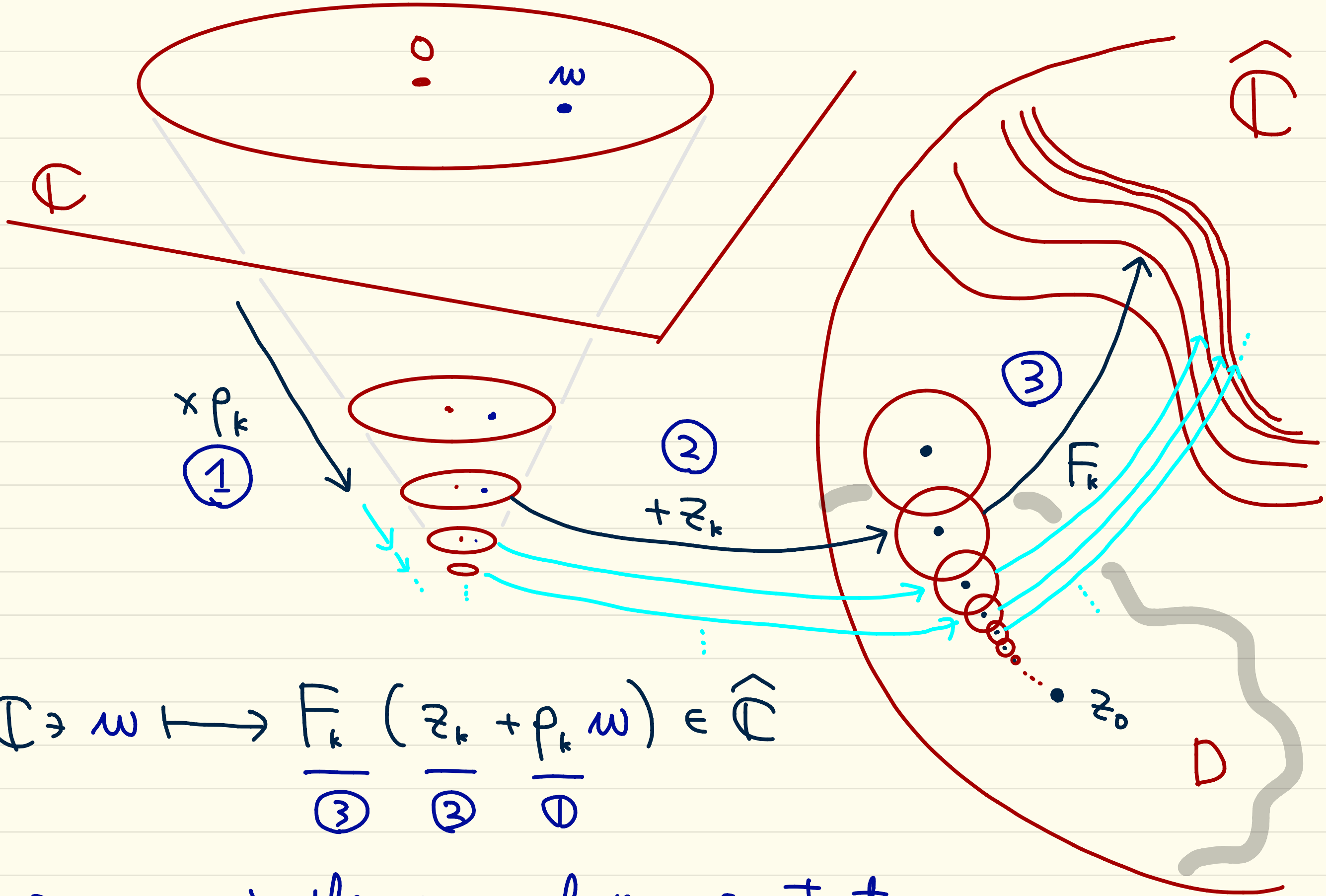
$\mathcal{F}$ is NOT normal at $z_0 \in D$

$\Longleftrightarrow$

$$\left.\begin{array}{l} \exists \{F_k\}_k \subset \mathcal{F} \\ \exists \{z_k\}_k \subset D \\ \exists \{\rho_k\}_k \subset \mathbb{C}^* \end{array}\right\} \text{s.t.} \quad \begin{array}{l} \text{as } k \to \infty, \; z_k \longrightarrow z_0, \; \rho_k \longrightarrow 0 \\ \text{and} \qquad \exists \\ F_k(z_k + \rho_k w) \rightrightarrows \phi(w) \end{array}$$

$\forall$ cpt sets in $\mathbb{C}$

non-const. mero. func. on $\mathbb{C}$



$$\mathbb{C} \ni w \longmapsto \underbrace{F_k}_{3}\,(\underbrace{z_k}_{2} + \underbrace{\rho_k}_{1}\, w) \in \hat{\mathbb{C}}$$

converges in the space of non-constant meromorphic functions on $\mathbb{C}$

Another way to understand:

$$\underbrace{T_k(w)}_{\text{cpx affine}} := z_k + \rho_k w \xrightarrow{} \underbrace{z_0}_{\text{const.}} \quad \begin{pmatrix} \text{as } k \to \infty \\ z_k \to z_0 \\ 0 \neq \rho_k \to 0 \end{pmatrix}$$

unif. conv. on $^\forall$cpt subsets in $\mathbb{C}$

Notation

Aff : the set of complex affine maps

$\subsetneq$ $\mathcal{U}$: the set of non-constant meromorphic functions on $\mathbb{C}$

$\mathcal{F}$ is NOT normal at $z_0 \in D$

$$\Longleftrightarrow \left.\begin{array}{l} ^\exists \{T_k\} \subset \mathrm{Aff} \\ ^\exists \{F_k\} \subset \mathcal{F} \end{array}\right\} \text{ s.t. } \left\{\begin{array}{l} T_k \longrightarrow z_0 : \text{constant map} \\ F_k \circ T_k \longrightarrow \phi \in \mathcal{U} \end{array}\right.$$

2 Dynamical Zalcman Functions (after Steinmetz)

$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a rational map of deg. $d \geqq 2$

Apply Zalcman's Lemma to $\mathcal{F} = \{f^n\}_{n \in \mathbb{N}}$

$\infty \neq z_0 \in J(f)$: the Julia set

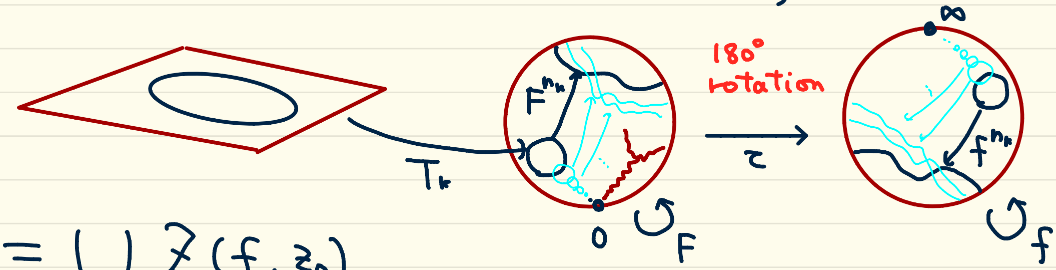
$$\iff \begin{array}{l} \exists \{T_k\} \subset \mathrm{Aff} \\ \exists \{n_k\} \subset \mathbb{N} \end{array} \Bigg\} \text{ s.t. } \begin{cases} T_k \longrightarrow z_0 & \text{: const.} \\ f^{n_k} \circ T_k \longrightarrow \exists \phi \in \mathcal{U} & \text{non-const. mero. on } \mathbb{C} \end{cases}$$

We say ϕ is a (dynamical) Zalcman function of f at z_0

$Z(f, z_0)$: the set of all possible $\phi \in \mathcal{U}$ as above.

When $z_0 = \infty \in J(f)$: $F := \tau \circ f \circ \tau^{-1}$ $\quad (\tau(z) = 1/z)$

$$\mathcal{Z}(f, \infty) := \{ \tau \circ \phi \in \mathcal{U} \mid \phi \in \mathcal{Z}(F, 0) \}$$



$$\mathcal{Z}(f) := \bigcup_{z_0 \in J(f)} \mathcal{Z}(f, z_0)$$

Example z_0: a repelling periodic point in $\mathbb{C}$ of period p

$$\lambda_0 := Df^p(z_0)$$

$$\phi(w) := \lim_{k \to \infty} f^{n_k}(z_0 + \lambda_0^{-k} w) \ : \text{Poincaré function}$$

$$\phi \circ f(w) = \phi(\lambda_0 w)$$

Invariance

$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ rational,

$\delta : \mathbb{C} \to \mathbb{C}$ complex affine ($\delta \in \mathrm{Aff}$)

$\phi : \mathbb{C} \longrightarrow \hat{\mathbb{C}}$ non-const. mero. ($\phi \in \mathcal{U}$)

$\Longrightarrow \quad f \circ \phi \in \mathcal{U} \quad \& \quad \phi \circ \delta \in \mathcal{U}$

$\Longrightarrow \quad f \circ \mathcal{U} \subset \mathcal{U} \quad \& \quad \mathcal{U} \circ \mathrm{Aff} = \mathcal{U}$

$\mathcal{Z}(f)$ has better invariance than $\mathcal{U}$:

Prop (Steinmetz)

$\forall\, z_0 \in J(f)$, $\quad f \circ \mathcal{Z}(f, z_0) = \mathcal{Z}(f, z_0) = \mathcal{Z}(f, z_0) \circ \mathrm{Aff}$

Hence $\quad f \circ \mathcal{Z}(f) = \mathcal{Z}(f) = \mathcal{Z}(f) \circ \mathrm{Aff}$

3 Parametric Zalcman functions

$D \subset \mathbb{C}$: a domain

$\mathfrak{f} := \{f_t\}_{t\in D}$: a holomorphic family of rat. maps of fixed degree $d \geq 2$.

$S(\mathfrak{f})$: the set of J-stable parameters

i.e. $t \in S(\mathfrak{f}) :\Longleftrightarrow \exists$ a nbd. U of t s.t. $\forall t' \in U$ f_t & $f_{t'}$ are qc'ly conjugate on their Julia sets

$B(\mathfrak{f}) := D - S(\mathfrak{f})$ the bifurcation locus of $\mathfrak{f}$

Normality and bifurcation

FACT $t_0 \in B(\mathbb{f}) \iff f_{t_0}$ has an active critical point

DEF A critical point g_0 of f_{t_0} is active

$:\iff \exists\, \nu \in \mathbb{N},\ \exists\, \delta > 0,\ \exists\, g : \mathbb{D} \longrightarrow \hat{\mathbb{C}}$ meromorphic

s.t. ∘ $\forall s \in \mathbb{D},\quad t := t_0 + \delta s^\nu \in \mathbb{D}$

∘ $\forall s \in \mathbb{D},\quad g(s)$ is a crit. pt. of f_t

∘ The family

$$\{ s \longmapsto f_t^n(g(s)) \}_{n \geq 0}$$

is NOT normal at $s = 0$ $(t = t_0)$

Example $\mathbb{f} = \{ z^3 - 3tz \}_{t \in \mathbb{C}}$ $\quad \begin{cases} t = s^2 \\ g(s) = \pm s \end{cases}$

$t_0 = 0 \notin B(\mathbb{f})$

Apply Zalcman's Lemma:

The family $\{ s \longmapsto f_t^n(g(s)) =: F_n(s) \}_{n \geq 0}$ is NOT normal at $s = 0$ $(t = t_0)$

$$\Longleftrightarrow \begin{matrix} \exists \{T_k\} \subset \mathrm{Aff} \\ \exists \{n_k\} \subset \mathbb{N} \end{matrix} \text{ s.t. } \begin{cases} T_k \longrightarrow 0 & : \text{constant} \\ F_{n_k} \circ T_k \longrightarrow \exists\, \Phi \in \mathcal{U} & \end{cases}$$

non-const. mero. on $\mathbb{C}$

We say $\Phi \in \mathcal{U}$ is a parametric Zalcman function of $\mathbb{f}$ at t_0

$\mathcal{P}(\mathbb{f}, t_0)$: the set of all possible Φ as above.

$$\mathcal{P}(\mathbb{f}) := \bigcup_{t_0 \in B(\mathbb{f})} \mathcal{P}(\mathbb{f}, t_0)$$

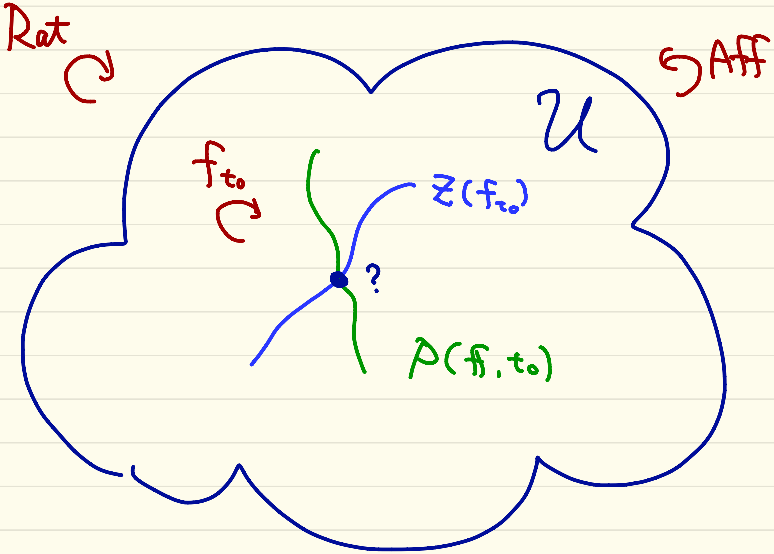
Invariance

Prop $\forall t_0 \in B(\mathfrak{f})$,

$$f_{t_0} \circ \mathcal{P}(\mathfrak{f}, t_0) = \mathcal{P}(\mathfrak{f}, t_0) = \mathcal{P}(\mathfrak{f}, t_0) \circ \mathrm{Aff}$$

and

$$\mathcal{P}(\mathfrak{f}) = \mathcal{P}(\mathfrak{f}) \circ \mathrm{Aff}$$



Question

Do $\mathcal{Z}(f_{t_0})$ and $\mathcal{P}(\mathfrak{f}, t_0)$ intersect ?

Any applications ?

4 Intersections in the quadratic family

$\mathbb{f} := \{ f_t(z) = z^2 + t \}_{t \in \mathbb{C}}$

critical points : $\underbrace{z_0 \equiv 0}_{\text{active?}}$ & $z_\infty \equiv \infty$ non-active (super attracting)

$$t_0 \in B(\mathbb{f}) \iff z_0 \text{ is an active crit. pt. of } f_{t_0}$$

$$\iff \{ s \longmapsto f_t^n(0) \}_{n \geq 0} \text{ is not normal}$$

$$\text{at } s = 0, \text{ where } t = t_0 + s$$

$$\iff \{ t \longmapsto f_t^n(0) \}_{n \geq 0} \text{ is not normal}$$

$$\text{at } t = t_0$$

$$\iff t_0 \in \partial \mathbb{M} \quad (\mathbb{M} : \text{the Mandelbrot set})$$

Semi-hyperbolic parameters

$t_0 \in \partial \mathbb{M}$ is semi-hyperbolic

$:\Longleftrightarrow$ $\gamma_0 = 0$ is not recurrent under f_{t_0}

and f_{t_0} has no parabolic cycles.

Example

t_0 : Misiurewicz $\Rightarrow$ semi-hyp.

(i.e. $\gamma_0 = 0$ lands on a repelling cycle)

Thm (K)

$t_0 \in \partial \mathbb{M} = B(\mathbf{f})$: semi-hyp.

$$\Longrightarrow \ {}^\exists \phi \in Z(f_{t_0}, \underset{J(f_{t_0})}{\underset{\Uparrow}{t_0}}) \cap \mathcal{P}(\mathbf{f}, t_0)$$

Lemma (k) $t_0 \in \partial\mathbb{M}$ semi-hyperbolic

$$\Longrightarrow \begin{array}{l} \exists \{n_k\} \subset \mathbb{N} \\ \exists \{\rho_k\} \subset \mathbb{C}^* \end{array} \text{ s.t. } \begin{cases} n_k \longrightarrow \infty & \text{as } k \to \infty \\ \rho_k \longrightarrow 0 \end{cases}$$

& $\exists\ \phi \in \mathcal{U}$, $\exists\ Q \in \mathbb{C}^*$ s.t.

(1) $\phi_k(w) := f_{t_0}^{n_k}(t_0 + \rho_k w) \longrightarrow \phi(w)$ in $\mathcal{U}$

(2) $\Phi_k(w) := f_{t_0+\rho_k Q w}^{n_k}(t_0 + \rho_k Q w) \longrightarrow \phi(w)$

Hence $\phi \in \mathcal{Z}(f_{t_0}, t_0)$ and $\phi \in \mathcal{D}(f, t_0)$.

§ Similarity between J, M & T

Hausdorff distance in $\hat{\mathbb{C}}$

$X, Y \subset \hat{\mathbb{C}}$ non-empty, compact

$$d_H(X, Y) := \inf\{\varepsilon > 0 \mid \underbrace{N_\varepsilon(Y)}_{\varepsilon\text{-open nbd of } Y} \supset X \ \&\ N_\varepsilon(X) \supset Y\}$$

w.r.t. spherical metric

Thm (Tan, Rivera-Letelier, K)

$\forall\, t_0 \in \partial \mathbb{M}$, semi-hyp. $\exists\, \{\rho_k\} \subset \mathbb{C}^*$ with $\rho_k \to 0$

$\exists\, Q \in \mathbb{C}^*$

s.t. for $R_k(z) := \rho_k^{-1}(z - t_0)$, $\hat{R}_k(z) := \rho_k^{-1} Q^{-1}(z - t_0)$

$$d_H\Big(R_k(J(f_{t_0})),\ \hat{R}_k(\mathbb{M})\Big) \longrightarrow 0$$

as $k \to \infty$

Idea

$$\begin{cases} z \notin J(f_{t_0}) \iff \exists n \text{ s.t. } |f_{t_0}^n(z)| > 2 \\ t \notin M \iff \exists n \text{ s.t. } |f_t^n(0)| > 2 \end{cases}$$

Take n_k, ρ_k, Q, ϕ as in the Lemma.

$$R_k^{-1}(\underset{\text{fixed}}{\omega}) = t_0 + \rho_k \omega \notin J(f_{t_0}) \text{ for } k \gg 0.$$

$$\Longrightarrow \quad \exists N \in \mathbb{N}, \ |f_{t_0}^{N+n_k}(t_0 + \rho_k \omega)| > 2$$

$$\Longrightarrow \quad \therefore \ |f_{t_0}^{N} \circ \phi(\omega)| > 2 \qquad (\phi \in \mathcal{U})$$

$$\Longrightarrow \quad \therefore \ |f_{t_0+\rho_k Q\omega}^{N} \circ f_{t_0+\rho_k Q\omega}^{n_k}(t_0 + \rho_k Q \omega)| > 2$$

$$\Longrightarrow \quad \therefore \ |f_{t_0+\rho_k Q\omega}^{N+n_k+1}(0)| > 2$$

$$\Longrightarrow \quad t_0 + \rho_k Q \omega \notin M \iff \hat{R}_k^{-1}(\omega) \notin M$$

This roughly explains "$N_\varepsilon(R_k(J)) \supset \hat{R}_k(M)$"

Tricorn (Crowe-Hasson-Rippon-Strain-Clark, Milnor)

$\mathcal{G} := \{ g_t(z) = \bar{z}^2 + t \}_{t \in \mathbb{C}}$ anti-holomorphic g_t^2 : holo.

$\mathbb{T} := \{ t \in \mathbb{C} \mid \forall n, \ |g_t^n(0)| \leqq 2 \}$

Rem. $\mathbb{T} \cap \mathbb{R} = \mathbb{M} \cap \mathbb{R} = [-2, 1/4]$

We say $t_0 \in \partial \mathbb{T}$ is Misiurewicz if 0 is pre-repelling.

Lemma (with a help by H. Inou) t_0 : Misiurewicz.

$\Longrightarrow \exists \{n_k\}, \exists \{p_k\}, \exists H : \mathbb{C} \to \mathbb{C}$ $\mathbb{R}$-linear, $\exists \phi \in \mathcal{U}$

(1) $g_{t_0}^{n_k}(t_0 + p_k w) \longrightarrow \phi(w)$ in $\mathcal{U}$

(2) $g_{t_0 + H(p_k w)}^{n_k}(t_0 + H(p_k w)) \longrightarrow \phi(w)$

$H(w) = Qw + Q'\bar{w}$

$|Q| + |Q'| \neq 0$

Thm $t_0 \in \partial \mathbb{T}$: Misiurewicz

$\exists \{\rho_k\} \subset \mathbb{C}^*$, $\exists H : \mathbb{C} \to \mathbb{C}$ $\mathbb{R}$-linear

$$R_k(z) := \rho_k^{-1}(z - c_0) \qquad \hat{R}_k(z) := \rho_k^{-1} H^{-1}(z - c_0)$$

$$d_H\left(R_k(J(g_{t_0})), \hat{R}_k(\mathbb{T}) \right) \longrightarrow 0$$

$\longrightarrow$ Movies!

THANK YOU!

Reference

T. Kawahira. Quatre applications du lemme de Zalcman à la dynamique complexe.

J. d'Anal. Math. 2014, pp 309-336

Movies available at :

http://www.math.titech.ac.jp/~kawahira/gallery.html