Dynamics of semigroups of transcendental entire functions

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A family of transcendental entire functions $\{f_1, f_2, \dots\}$.

$$G = \langle f_1, f_2, \dots \rangle$$

The Fatou set of G is the maximal open subset of $\mathbb{C}$ where the family of G is normal. Denote it by $F(G)$.

The Julia set of G is $J(G) = \mathbb{C} \setminus F(G)$.

- For each $g \in G$,

$$\begin{aligned} F(G) &\subset F(\langle g \rangle) \\ J(\langle g \rangle) &\subset J(G) \end{aligned}$$

- The Fatou set is forward invariant
- The Julia set is backward invariant.
- Julia sets may have interior points.

Theorem. (Stankewitz) *The Julia set of a semigroup of entire functions which contains at least one transcendental entire function is the closure of the set of repelling fixed points.*

Corollary.

$$J(G) = \overline{\bigcup_{f \in G} J(\langle f \rangle)}$$

Let U be a component of $F(G)$.

For $g \in G$, U_g is a component of $F(G)$ satisfying $g(U) \subset U_g$.

$$G_U = \{g \in G \mid U_g = U\}$$

U is a stable domain, if $G_U \neq \emptyset$.

- (super-)attracting, if U is a subdomain of (super-) attracting basin of some element of G_U.
- parabolic, if U is a subdomain of parabolic basin of some element of G_U.
- Siegel, if U is a subdomain of Siegel disk of some element of G_U.
- Baker, if $\exists \{g_n\} \subset G_U$ such that $g_n \to \infty$ l.u. on U.

$U \subset F(G)$ is wandering, if $\#\{U_g \mid g \in G\} = \infty$.

U is strictly wandering, if $U_g = U_h \Rightarrow g = h$.

Theorem.(Hinkkanen and Martin) *There exists an infinitely generated polynomial semigroup which has wandering domains.*

Conjecture.(Hinkkanen and Martin) *Let G be a finitely generated polynomial semigroup. Then G has no wandering domainss.*

$$\mathrm{sing}(f^{-1}) = \{\text{critical values of } f\} \cup \{\text{asymptotic values of } f\}$$

$$\mathcal{S} = \{f \mid \#\mathrm{sing}(f^{-1}) \text{ is finite.}\}$$

$$\mathcal{B} = \{f \mid \mathrm{sing}(f^{-1}) \text{ is bounded.}\}$$

Theorem.(Eremenko and Lyubich) *If* $f \in \mathcal{B}$, *then for* $\forall z \in F(\langle f \rangle)$ $f^n(z) \not\to \infty$.

Theorem. *For* $f \in \mathcal{B}$, *every component of* $F(\langle f \rangle)$ *is simply connected.*

Proposition. *If* G *is generated by functions belonging to* $\mathcal{B}$, *then every component of* $F(G)$ *is simply connected.*

Theorem. $f \in \mathcal{S}$ *has neither Baker domains nor wandering domains.*

Theorem.(Bishop) *There is a function belonging to $\mathcal{B}$ whose Fatou set contains wandering domains.*

Theorem.(Fagella, Godillon and Jarque) *There are $f, g \in \mathcal{B}$ such that neither $F(\langle f \rangle)$ nor $F(\langle g \rangle)$ has wandering domains and $F(\langle f \circ g \rangle)$ has wandering domains.*

Question. *Does $G = \langle f, g \rangle$ have wandering domains?*

Question. *Let G be a finitely generated semigroup of functions belonging to $\mathcal{B}$. Does G have wandering domains?*

Question. *Let G be a finitely generated semigroup of functions belonging to $\mathcal{B}$. Does G have Baker domains?*

Question. *Let G be a finitely generated semigroup of functions belonging to $\mathcal{S}$. Does G have wandering domains?*

Question. *Let G be a finitely generated semigroup of functions belonging to $\mathcal{S}$. Does G have Baker domains?*

Example. $f(z) = e^z - 1$, $g(z) = 2z$, $G = \langle f, g \rangle$

$J(\langle g \rangle) = \{0\}$

$\{z \mid \text{Re } z < 0\} \subset F(\langle g \rangle)$

$g(\{z \mid \text{Re } z < 0\}) \subset \{z \mid \text{Re } z < 0\}$

0 is a parabolic fixed point of f.

$\{z \mid \text{Re } z < 0\} \subset F(\langle f \rangle)$

$f(\{z \mid \text{Re } z < 0\}) \subset \{z \mid \text{Re } z < 0\}$

$\{z \mid \text{Re } z < 0\} \subset F(G)$

For $n \in \mathbb{Z}$,

$$\{x + 2\pi n i \mid x \geq 0\} \subset J(\langle f \rangle)$$

$$\left\{ x + 2\pi \frac{n}{2^m} i \middle| \, x \geq 0 \right\} = g^{-m}(\{x + 2\pi n i \mid x \geq 0\}) \subset J(G)$$

$$\{z \mid \mathrm{Re}\ z \geq 0\} \subset J(G)$$

$$\{z \mid \mathrm{Re}\ z < 0\} = F(G), \qquad \{z \mid \mathrm{Re}\ z \geq 0\} = J(G)$$

$\{z \mid \mathrm{Re}\ z < 0\}$ is a Baker domain.

$\{g^n\}$

$\{g^n \circ f\}$

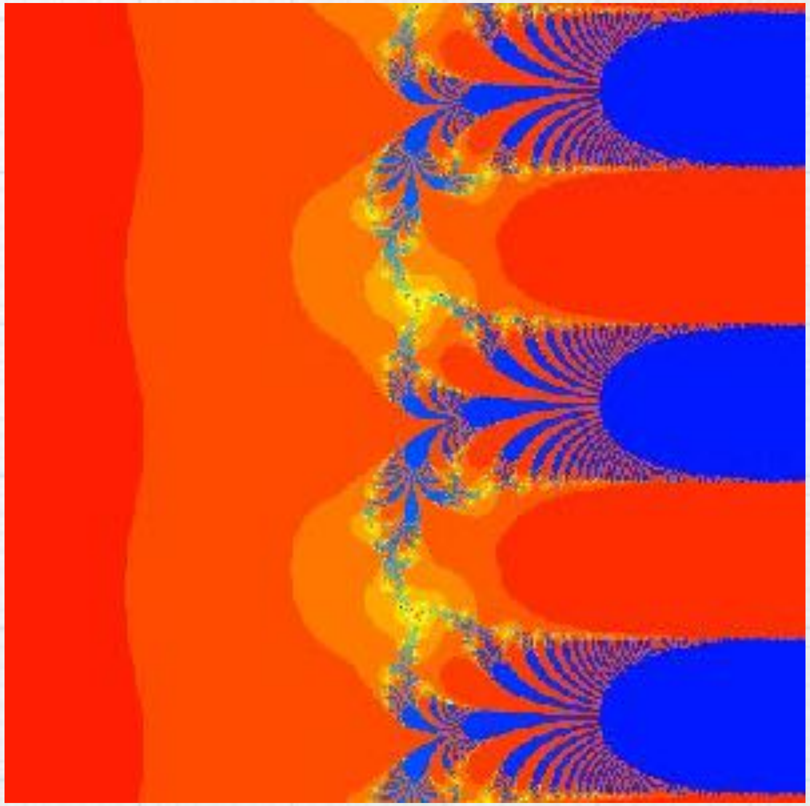
Example.

$$f(z) = e^z + k$$

$$J(\langle f \rangle) = J(\langle f \rangle) + 2\pi i$$

$$g(z) = e^z + k + 2\pi n i$$

$$J(\langle f \rangle) = J(\langle g \rangle)$$



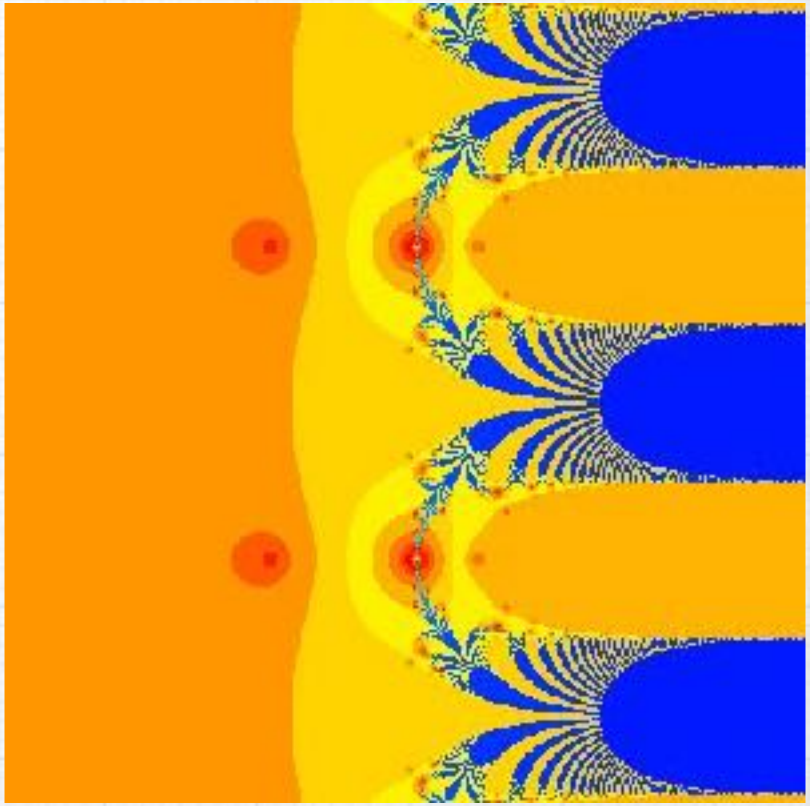
$$f_0(z) = e^z + c + \pi i, \qquad c > 1$$

$f_0(z)$ has an attracting 2-cycle.

$$\alpha < 0 < \beta$$

$$f_0(\alpha + \pi i) = \beta + \pi i$$

$$f_0(\beta + \pi i) = \alpha + \pi i$$



$$\begin{aligned} f_n(z) &= e^z + c + (2n+1)\pi i \\ &= f_0(z) + 2n\pi i \end{aligned}$$

$$\{\alpha + (2n+1)\pi i, \beta + (2n+1)\pi i\}$$

$$J(\langle f_0 \rangle) = J(\langle f_n \rangle)$$

$$G = \langle \cdots, f_{-1}, f_0, f_1, \cdots \rangle$$

$$J(G) = J(\langle f_0 \rangle) \qquad F(G) = F(\langle f_0 \rangle)$$

A is the component of $F(G)$ intersecting the left half-plane.

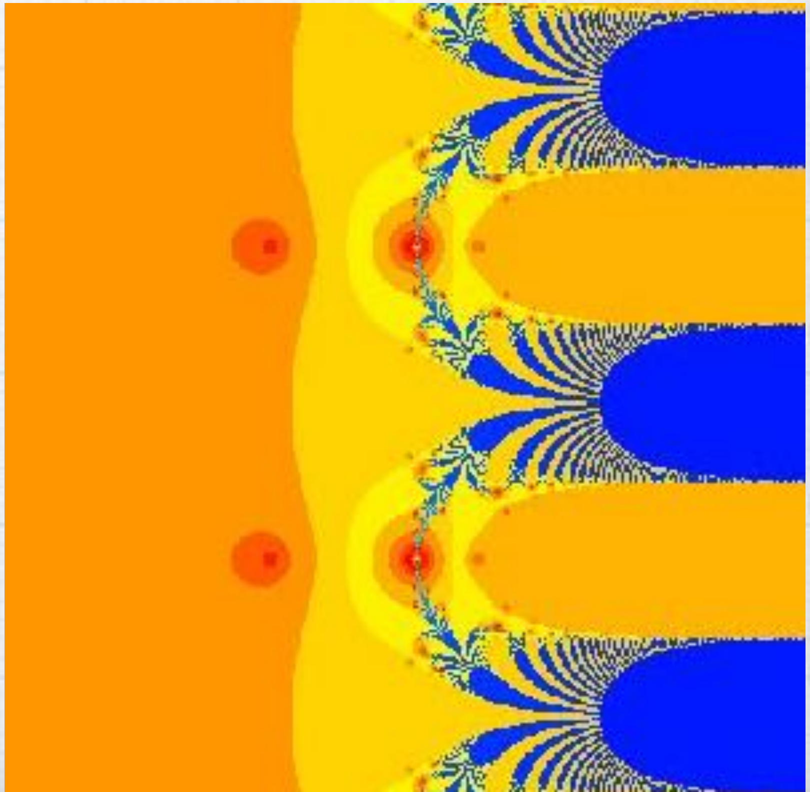
B_m is the component of $F(G)$ containing $\beta + (2m + 1)\pi i$.

$$f_n(A) = B_n \qquad f_n(B_m) = A$$

$$\{f_{n+1} \circ f_n(z) = f_0^2(z) + 2(n+1)\pi i\}$$

B_m is a wandering domain.

A is a Baker domain.



- G has wandering domains.
- G has a Baker domain.

A semigroup G is nearly abelian if there is a compact family of Möbius transformations $\Phi = \{\phi\}$ satisfying:
(1) $\phi(F(G)) = F(G)$ for $\forall \phi \in \Phi$ and
(2) for $\forall f, g \in G$, $\exists \phi \in \Phi$ such that $f \circ g = \phi \circ g \circ f$.

Theorem. *Let G be a nearly abelian transcendental entire semigroup. Then for each $g \in G$, $J(G) = J(\langle g \rangle)$.*

$$f_1(z) = e^z + k = e^z + k + 2\pi n_1 \pi i$$

$$f_j(z) = e^z + k + 2\pi n_j \pi i \qquad (2 \le j \le m)$$

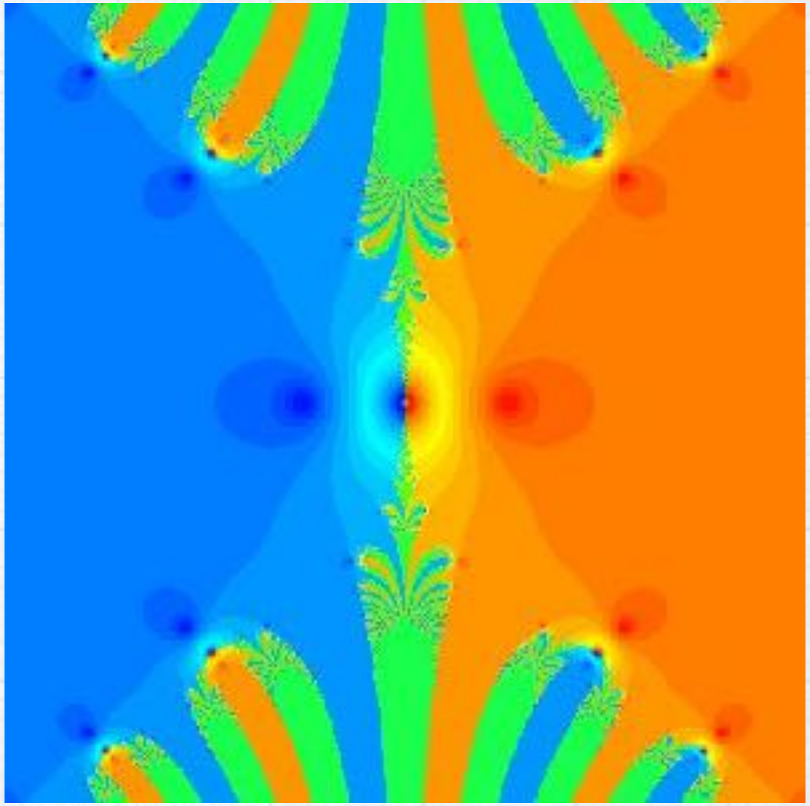
$$f_{s_\ell} \circ f_{s_{\ell-1}} \circ \cdots \circ f_{s_1}(z) = f_1^{\ell}(z) + 2\pi n_{s_\ell} i$$

$\langle f_1, f_2, \cdots f_n \rangle$ is a nearly abelian semigroup.

$a > 1$

$$g_1(z) = a \int_0^z e^{-w^2} dz$$

$$g_2(z) = -a \int_0^z e^{-w^2} dz$$



$\langle g_1, g_2 \rangle$ is a nearly abelian semigroup.

Question. *What can we say about a nearly abelian transcendental entire semigroup?*

Question. *Does a nearly abelian transcendental entire semigroup have wandering domains?*