On formal normal forms of holomorphic germs at super-saddle fixed points

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Contents

§1. Regular polynomial skew products on $\mathbb{C}^2$

$$f(z,w) = (p(z), q(z,w))$$
$$= \left(z^d + \sum_{j=0}^{d-1} a_j z^j,\ \ w^d + \sum_{i+j\le d,\, j<d} b_{i,j} z^i w^j \right) (*)$$
$$f : \{z\} \times \mathbb{C} \to \{p(z)\} \times \mathbb{C}.$$

f preserves the family of *fibers* $\{z\} \times \mathbb{C}$.

$$q_z(w) := q(z,w),$$
$$f^k(z,w) = (p^k(z), Q_z^k(w)) := (p^k(z), q_{p^{k-1}(z)} \circ \cdots \circ q_z(w)).$$

$$K_p := \{z \in \mathbb{C}; \{p^k(z)\}_{k \geq 0} \text{ is bounded}\} \text{ (filled Julia set)}$$

$$J_p := \partial K_p \text{ (Julia set)}$$

For $z \in K_p$,

$$K_z := \{w \in \mathbb{C}; \{Q_z^k(w)\}_{k \geq 0} \text{ is bounded}\},$$

$$J_z := \partial K_z \ \text{(}\textit{fiber Julia set}\text{)}.$$

§2. Continuity of the map $z \mapsto J_z$ on K_p for Axiom A map

$$\begin{aligned}
A_p &:= \{\text{attracting periodic points of } p \text{ in } \mathbb{C}\},\\
\Lambda_{A_p} &:= \overline{\{\text{saddle periodic points of } f \text{ in } A_p \times \mathbb{C}\}},\\
\Lambda_{J_p} &:= \overline{\{\text{saddle periodic points of } f \text{ in } J_p \times \mathbb{C}\}},\\
W^s(\Lambda_{A_p}) &:= \{y \in \mathbb{C}; f^n(y) \to \Lambda_{A_p}\},\\
W^u(\Lambda_{J_p}) &:= \{y \in \mathbb{C}; \exists \text{ prehistory } \{y_{-k}\} \to \Lambda_{J_p}\}.
\end{aligned}$$

Theorem 1. *Suppose f is Axiom A. Then*

$$z \mapsto J_z \textit{ is continuous on } K_p \iff W^s(\Lambda_{A_p}) \cap W^u(\Lambda_{J_p}) = \emptyset.$$

Example with $W^s(\Lambda_{A_p}) \cap W^u(\Lambda_{J_p}) \neq \emptyset$

$$f(z,w) = (p(z), q_z(w)) = (z^2 + az, w^2 + 2(b-z)w),$$

$$|a| < 1, \quad |b| > \frac{1}{2}, \quad 0 < |a - 1 + b| < \frac{1}{2}.$$

$$A_p = \{0\}, \quad \alpha_+ = (0,0) \in \Lambda_{A_p}.$$

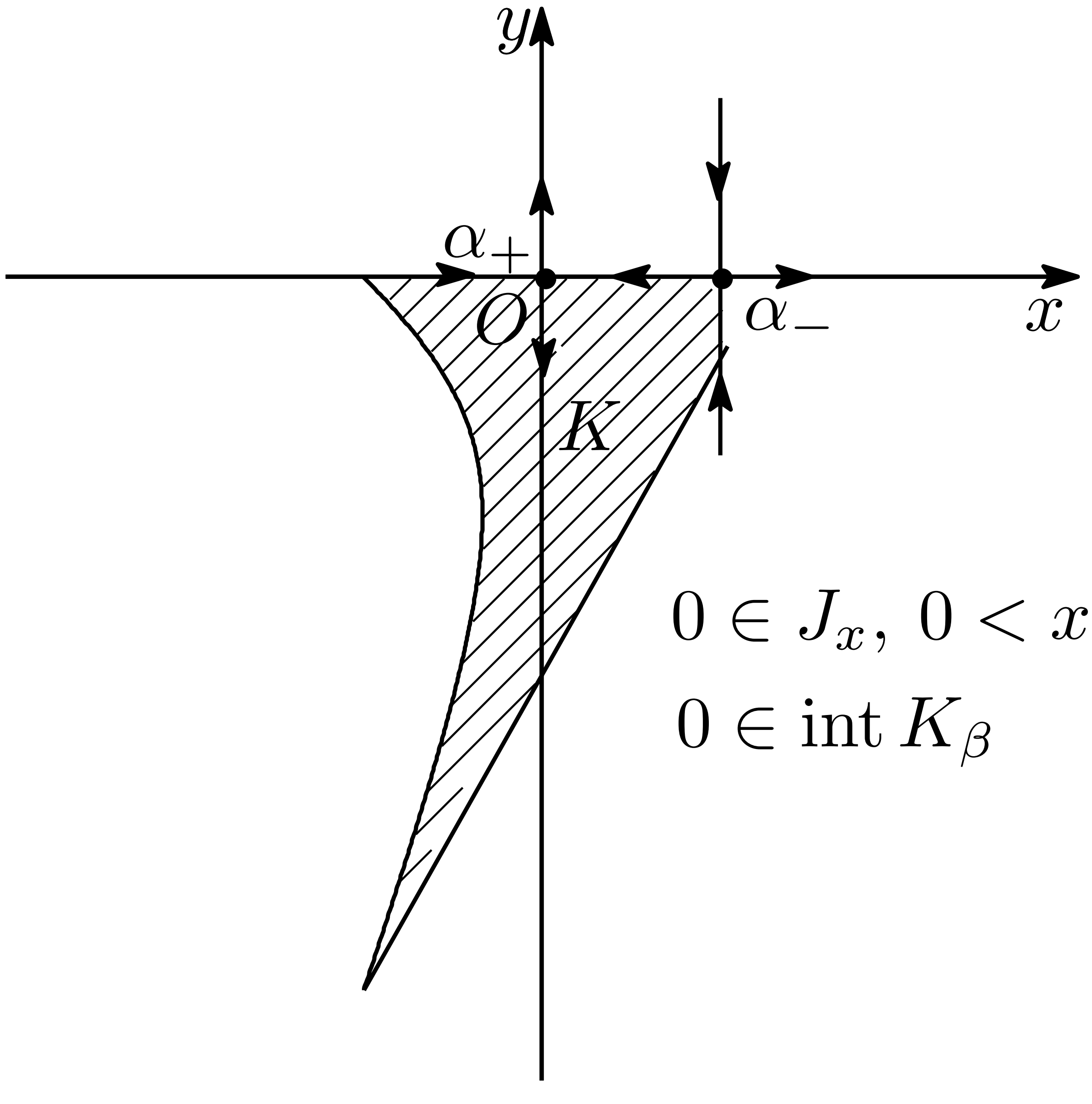
$$\alpha_- = (\beta, 0) := (1-a, 0) \in \Lambda_{J_p}$$

$$W^s(\alpha_+) \cap W^u(\hat{\alpha}_-) \supset int\, K_p \times \{0\}.$$

The map $f(z,w) = (z^2, w^2 + 2(1-z)w)$ is not Axiom A

(cf. M. Jonsson)

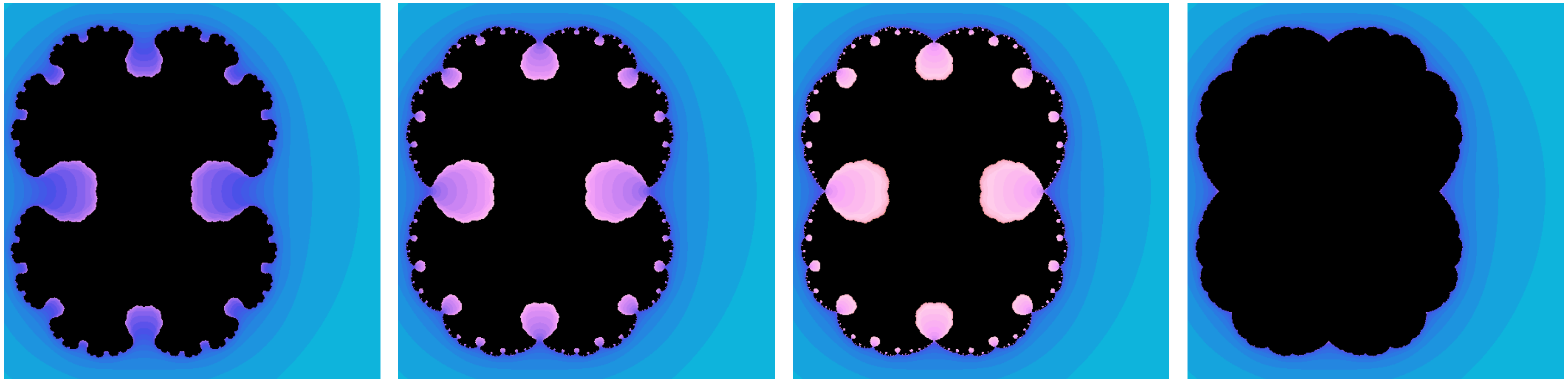
$W^u(\hat{\alpha}_-), W^s(\alpha_+)$ and the K-set in the real slice



$0 \in J_x,\ 0 < x < \beta$

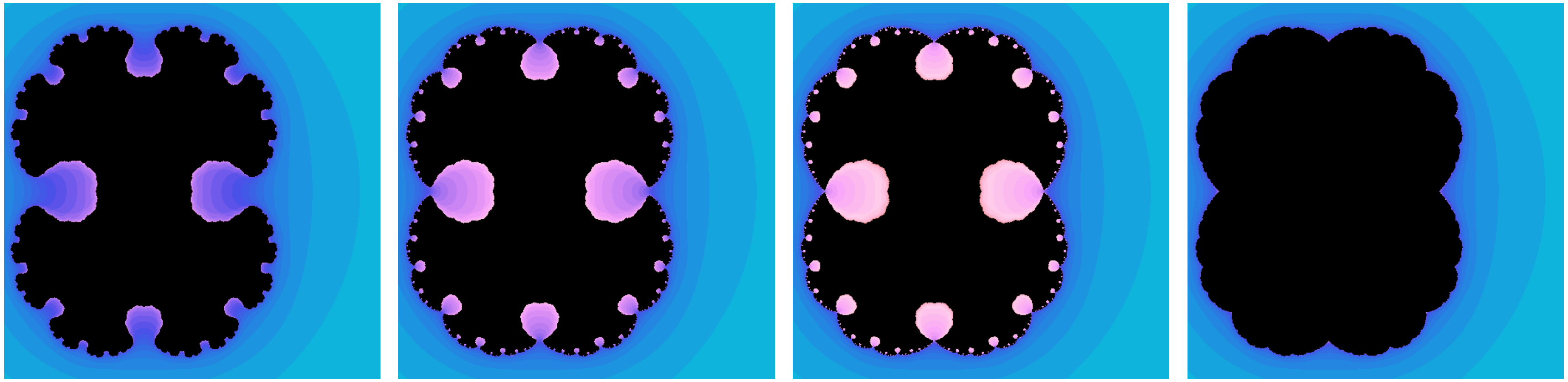
$0 \in \operatorname{int} K_\beta$

J_z for $f(z, w) = (z^2 + 0.1z, w^2 + 2(1.2 - z)w)$.



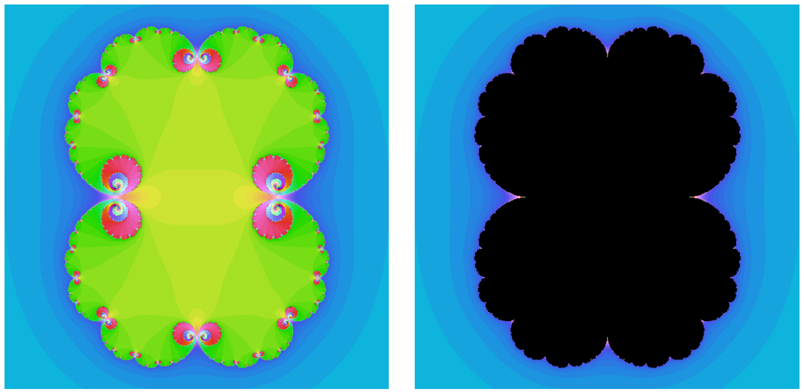
From left: $z = 0.89,\ 0.899,\ 0.8999,\ \beta = 0.9$

J_z for $f(z,w) = (z^2 + 0.1z, w^2 + 2(1.2 - z)w)$.

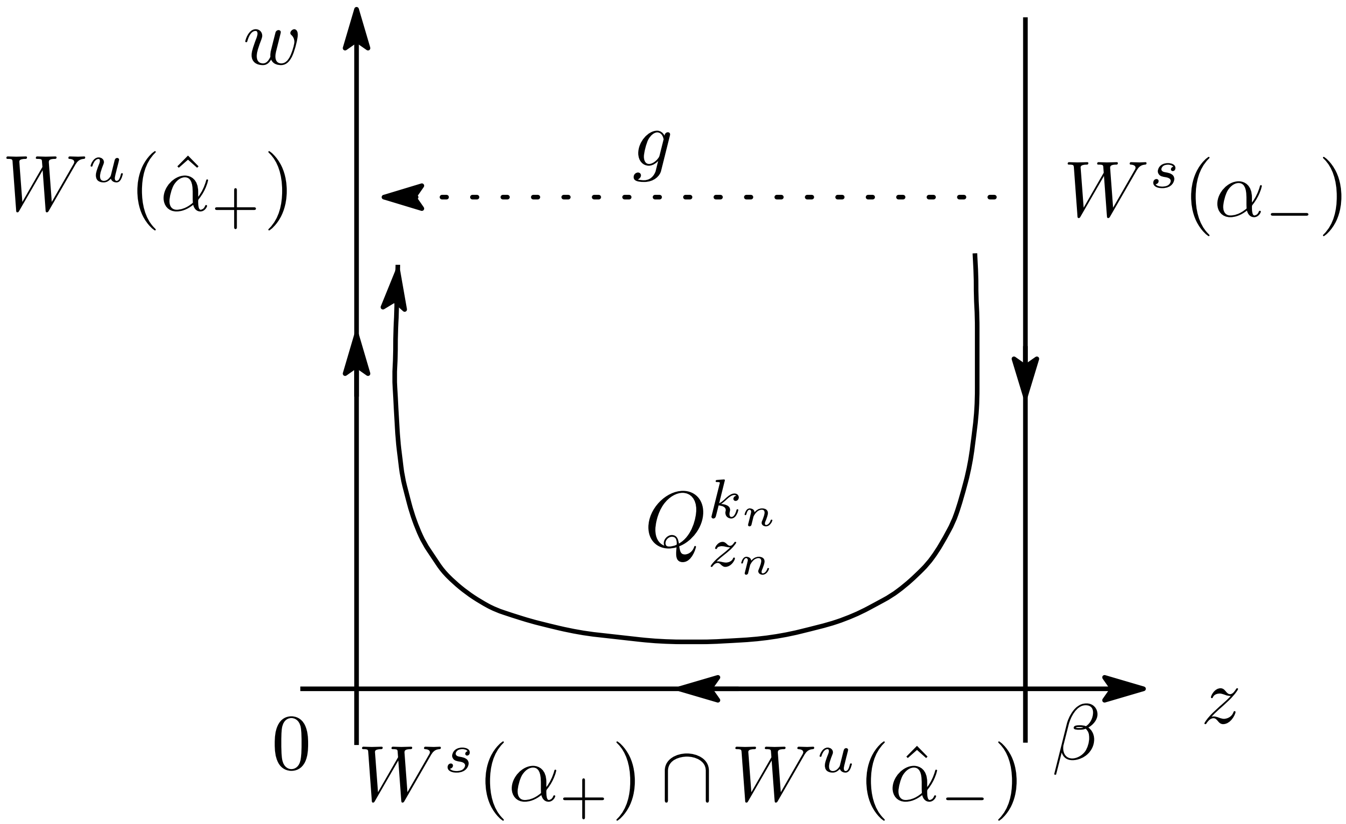


From left: $z = 0.89,\ 0.899,\ 0.8999,\ \beta = 0.9$

This phenomenon looks like parabolic implosion:



§3. An implosion arising from saddle connection (joint with H. Inou)



The base variable z plays the role of parameter.

The fiberwise dynamics drastically changes as z moves from β.

This causes the discontinuity of fiber Julia sets at $z = \beta$.

Under some assumptions, we will show:

For a sequence $z_n \in W^s(\alpha_+) \cap W^u(\hat{\alpha}_-)$ tending to β, there exist $k_n \to \infty$ such that the high iterates $Q_{z_n}^{k_n}$ converge to a "Lavaurs map" $g : W^s(\alpha_-) \to W^u(\hat{\alpha}_+)$.

We use linearizing coordinates at saddle fixed points instead of Fatou coordinates in parabolic implosion.

The family $\mathcal{F} = \mathcal{F}_d$ of maps $f(z, w) = (p(z), q_z(w))$ of the form $(*)$, satisfying

(1) p has an attracting fixed point 0, whose immediate basin U_0 contains a repelling fixed point β on ∂U_0.

(2) $q_z(0) \equiv 0$.

(3) $\alpha_+ = (0, 0)$ and $\alpha_- = (\beta, 0)$ are saddle fixed points:

$$Df(\alpha_\pm) = \begin{pmatrix} \lambda_\pm & 0 \\ 0 & \mu_\pm \end{pmatrix},$$

$$0 < |\lambda_+| < 1 < |\mu_+|, \qquad 0 < |\mu_-| < 1 < |\lambda_-|.$$

Linearization at saddle fixed points

The linearizing coordinates $\Psi_\pm$ at $\alpha_\pm$:

$$\Psi_\pm \circ f = Df(\alpha_\pm) \cdot \Psi_\pm, \quad D\Psi_\pm(\alpha_\pm) = Id.$$

Lemma 1. *There exists a subset $\mathcal{F}' \subset \mathcal{F}$ of full measure such that $\Psi_\pm$ uniquely exist for $f \in \mathcal{F}'$.*

By uniqueness, it follows that $\Psi_\pm$ are skew products :

$$\Psi_\pm(z, w) = (\phi_\pm(z), \psi_\pm(z, w)) = (\phi_\pm(z), \psi_{\pm,z}(w)).$$

$$\phi_\pm \circ p(z) = \lambda_\pm \phi_\pm(z),$$
$$\psi_{\pm,p(z)} \circ q_z = \mu_\pm \psi_{\pm,z}.$$

Take a sequence z_n in U_0, $z_n \to \beta$. Let

$$\mathcal{A}_\beta := \phi_-^{-1}(\{r \le |z| < |\lambda_-|r\}), \mathcal{A}_0 := \phi_+^{-1}(\{|\lambda_+|r < |z| \le r\})$$

be fixed fundamental domains of p around $\beta, 0$. Then

$$\exists k'_n \ (\to \infty) \text{ such that } p^{k'_n}(z_n) \in \mathcal{A}_\beta.$$

We assume

(4) $\overline{\{p^{k'_n}(z_n)\}}$ is compact in U_0.

(5) There exist $k''_n > k'_n$ such that $p^{k''_n}(z_n) \in \mathcal{A}_0$.

(6) $q'_z(0) = 0$ implies $q''_z(0) \neq 0$ for $z \in U_0$.

(7) $q'_{p^i(z)}(0) = 0$ implies $q'_{p^j(z)}(0) \neq 0$ for $j \neq i, z \in U_0$.

Decomposition of $Q_{z_n}^{k_n}$

$$
\begin{array}{ccccccc}
\mathbb{C},0 & \xleftarrow{\mu_+^{k_n - k_n''} w} & \mathbb{C},0 & \xleftarrow{\chi_n} & \mathbb{C},0 & \xleftarrow{\mu_-^{k_n'} w} & \mathbb{C},0 \\
\uparrow \psi_{+,p^{k_n}(z_n)} & & \uparrow \psi_{+,p^{k_n''}(z_n)} & & \uparrow \psi_{-,p^{k_n'}(z_n)} & & \uparrow \psi_{-,z_n} \\
\mathbb{C},0 & \xleftarrow[Q^{k_n - k_n''}_{p^{k_n''}(z_n)}]{} & \mathbb{C},0 & \xleftarrow[Q^{k_n'' - k_n'}_{p^{k_n'}(z_n)}]{} & \mathbb{C},0 & \xleftarrow[Q^{k_n'}_{z_n}]{} & \mathbb{C},0
\end{array}
$$

By the assumption (4) on $\{p^{k_n'}(z_n)\}$, the family of maps

$\chi_n := \psi_{+,p^{k_n''}(z_n)} \circ Q^{k_n''-k_n'}_{p^{k_n'}(z_n)} \circ \psi^{-1}_{-,p^{k_n'}(z_n)}$ is normal.

$Q_{z_n}^{k_n}$ in the linearizing coordinates

$$\psi_{+,p^{k_n}(z_n)} \circ Q_{z_n}^{k_n} \circ \psi_{-,z_n}^{-1}(w)$$

$$= \mu_+^{k_n - k''_n} \chi_n(\mu_-^{k'_n} w)$$

$$= \mu_+^{k_n - k''_n} \left(\chi'_n(0)\mu_-^{k'_n} w + \frac{1}{2}\chi''_n(0)\mu_-^{2k'_n} w^2 + O(\mu_-^{3k'_n}) \right)$$

$$\asymp (Q_{z_n}^{k_n})'(0) \left(w + \frac{\chi''_n(0)}{2} \frac{(Q_{z_n}^{k'_n})'(0)}{(Q_{p^{k'_n}(z_n)}^{k''_n - k'_n})'(0)} w^2 + O(\mu_-^{k'_n}) \right).$$

By the assumptions (6) and (7), the third term is negligible.

(6) and (7) hold if $d = 2$.

Convegence to Lavaurs maps

Theorem 2. *1) Suppose that*

$$\frac{(Q_{z_n}^{k'_n})'(0)}{(Q_{p^{k'_n}(z_n)}^{k''_n - k'_n})'(0)} \to 0.$$

If a sequence $\{k_n\}$ *satisfies* $(Q_{z_n}^{k_n})'(0) \to \sigma \neq 0$*, then*

$$Q_{z_n}^{k_n} \to g_\sigma := \psi_{+,0}^{-1} \circ m_\sigma \circ \psi_{-,\beta},$$

locally uniformly in $W^s(\alpha_-)$*, where* $m_\sigma(w) := \sigma w$*.*

2) Next suppose that

$$\left\{ \left| \frac{(Q_{z_n}^{k'_n})'(0)}{(Q_{p^{k'_n}(z_n)}^{k''_n - k'_n})'(0)} \right| \right\} \text{ is bounded from below.}$$

If a sequence $\{k_n (\geq k''_n)\}$ *satisfies*

$$(Q_{z_n}^{k_n})'(0) \to \sigma, \quad (Q_{z_n}^{k_n})''(0) \to \tilde{\tau} \neq 0, \text{ then}$$

$Q_{z_n}^{k_n} \to g_\eta := \psi_{+,0}^{-1} \circ \eta \circ \psi_{-,\beta}$, *locally uniformly in* $W^s(\alpha_-)$.

Here

$\eta(w) = \sigma w + \tau w^2, \; \tau = \tilde{\tau} + (\psi''_{+,0}(0)\sigma^2 - \psi'_{-,\beta}(0)\sigma)/2.$

Lavaurs maps and fiber Julia-Lavaurs sets

We call $g = g_\sigma$ or $g_\eta : W^s(\alpha_-) \to W^u(\hat{\alpha}_+)$ a *Lavaurs map*. Since

$$W^s(\alpha_-) = \{\beta\} \times int\, K_\beta \quad \text{and} \quad W^u(\tilde{\alpha}_+) = \{0\} \times \mathbb{C}$$

are disjoint, we cannot define the composition g^2 of g.

$$K(g) := K_\beta \setminus g^{-1}(\mathbb{C} \setminus K_0),$$
$$J(g) := \overline{g^{-1}(J_0)} \text{ (fiber Julia-Lavaurs set).}$$

It holds that

$$J_\beta \subsetneq J(g) = \partial K(g).$$

Fiber Julia sets accumulate fiber Julia-Lavaurs sets

Theorem 3. *Suppose that the assumptions in Theorem 2 hold, hence* $Q_{z_n}^{k_n} \to g$ *in* $W^s(\alpha_-)$*. We also assume that* q_0 *is hyperbolic. Then,*

$$\partial(K_{z_n}, K(g)) \to 0, \qquad \partial(J(g), J_{z_n}) \to 0.$$

If, in addition, $\operatorname{int} K_\beta = W^s(\alpha_-)$*, then, with respect to the Hausdorff distance,*

$$K_{z_n} \to K(g), \qquad J_{z_n} \to J(g).$$

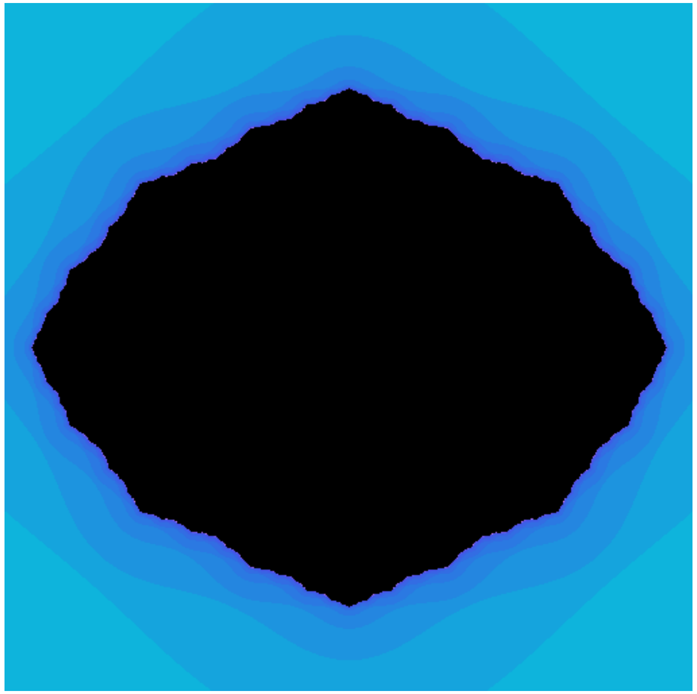
This explains the discontinuity of fiber Julia sets at $z = \beta$.

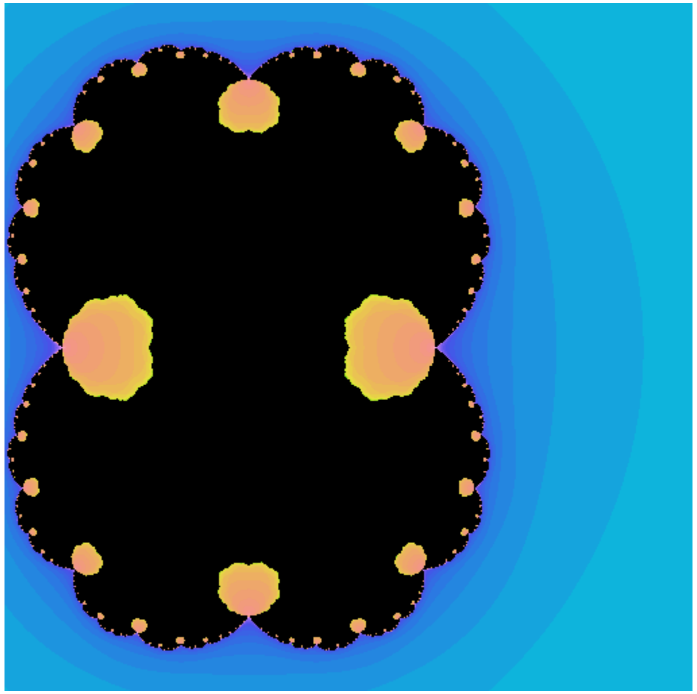
Topology of fiber Julia-Lavaurs sets

$f(z, w) = (z^2 + 0.1z, w^2 + 2(1.25 - z)w), \;\; g = g_\sigma, \; \sigma \in \mathbb{R}$.

$g^{-1}(J_0)$ consists of countably many arcs.

$J(g) = J_\beta \cup g^{-1}(J_0)$ is connected and locally connected.





§4. Super-saddle : $Spec(Df(\alpha)) = \{0, \mu\}, |\mu| > 1$

f : holomorphic germ $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$.

$\mathcal{C}(f)$: critical set of f and generalaized critical set :

$$\mathcal{C}^\infty(f) := \bigcup_{n \geq 0} f^{-n}(\mathcal{C}(f)) = \bigcup_{n \geq 0} \mathcal{C}(f^n).$$

f is rigid if

(a) $\mathcal{C}^\infty(f)$ has normal crossings, i.e., by a local coordinate,

$$\mathcal{C}^\infty(f) = \emptyset, \text{ or } \{z = 0\}, \text{ or } \{zw = 0\}.$$

(b) $f(\mathcal{C}^\infty(f)) \subset \mathcal{C}^\infty(f)$.

Formal normal forms of rigid germs

Favre classified attracting rigid germs and gave formal normal forms.
He showed formal classification coincides with analytic one in most cases.

Ruggiero extends his results to semi-superattracting rigid germs : $(Spec(Df(\alpha)) = \{0, \mu\}, \mu \neq 0)$.

He showed that, if $|\lambda| > 1$,

$$f(z, w) = (\lambda z, zw(1 + w)) \sim_{formal} f_0(z, w) = (\lambda z, zw),$$

but the formal conjugacy diverge.

In case $\alpha_+ = (0,0)$ is a super-saddle: $\lambda_+ = p'(0) = 0$

We may assume $p(z) = z^m$ with $m \geq 2$.

f is rigid at α_+, since $\mathcal{C}^\infty(f) = \{z = 0\}$.

By Ruggiero :

f is formally conjugate to the germ $f_0(z,w) = (z^m, \mu_+ w)$.

It turns out that, in most cases, the formal conjugacies diverge.

Divergence of formal conjugacies

$$f(z,w) = (z^m, q(z,w)), \quad m \geq 2,$$

where q is a polynomial of w whose coefficients are holomorphic at $z = 0$.

$$q(z,w) = \sum_{i \geq 0} q_i(w) z^i, \quad q_0(w) = \mu w + O(w^2), \quad |\mu| > 1.$$

ϕ_0 : the inverse of the linearizing coordinate of q_0 at 0 :

$$q_0 \circ \phi_0(w) = \phi_0(\mu w), \quad \phi_0'(0) = 1.$$

It extends to $\mathbb{C}$ by the functional relation :

$$\phi_0(w) = q_0^k \circ \phi_0(\frac{w}{\mu^k}).$$

Theorem 4. *Suppose that there exists* $w_0 \in \mathbb{C}$ *such that*

(i) $q_0'(\phi_0(w_0)) = 0$,

(ii) $q_1(\phi_0(w_0)) \neq 0$.

Then formal conjugacy $\Phi(z, w) = (z, \phi(z, w))$ *of* f *to* f_0 *is not holomorphic at the origin.*

The assumption (i) holds if and only if

$$q_0 \not\sim_{affine} r(w) := (w + 1)^d - 1, \text{ for any } d \geq 2.$$

In fact, for any neighborhood U of $0 \in J_{q_0}$,

$$\phi_0(\mathbb{C}) = \bigcup_{n \geq 0} \phi_0(\mu^n \cdot U) = \bigcup_{n \geq 0} q_0^n(\phi_0(U)) = \mathbb{C} \setminus \mathcal{E}(q_0),$$

where $\mathcal{E}(q_0)$ is the set of exceptional points of q_0.

$$\begin{aligned} q_0 \not\sim_{affine} r &\iff \mathcal{E}(q_0) = \{\infty\} \\ &\iff \phi_0(\mathbb{C}) = \mathbb{C} \\ &\iff \exists w_0 \in \mathbb{C}, \text{ s.t. (i).} \end{aligned}$$

If $q_0(w) = (w+1)^d - 1$, then $\phi_0(w) = e^w - 1$ and

$$q_0'(\phi_0(w)) = de^{(d-1)w} \neq 0.$$

Proof of Theorem 4

Formal conjugacy: $f \circ \Phi = \Phi \circ f_0$, $f_0(z, w) = (z^m, \mu w)$.

We may assume $\Phi(z, w) = (z, \phi(z, w))$. Then ϕ satisfies

$$q(z, \phi(z, w)) = \phi(z^m, \mu w).$$

Put $\phi(z, w) = \sum_{i \geq 0} \phi_i(w) z^i$, then

$q(z, \phi(z, w)) = \sum_{i \geq 0} q_i(\phi_0 + \sum_{j \geq 1} \phi_j z^j) z^i$

$= q_0(\phi_0) + q_0'(\phi_0) \sum_{j \geq 1} \phi_j z^j + \cdots$

$+ \left(q_1(\phi_0) + q_1'(\phi_0) \sum_{j \geq 1} \phi_j z^j + \cdots \right) z + \cdots$,

and $\phi(z^m, \lambda w) = \sum_{j \geq 0} \phi_j(\lambda w) z^{mj}$.

$$
\begin{aligned}
q_0(\phi_0(w)) &= \phi_0(\mu w), \\
q_0'(\phi_0)\phi_1(w) + q_1(\phi_0) &= 0, \\
&\cdots \\
q_0'(\phi_0)\phi_j(w) + F_j(\phi_0, \cdots, \phi_{j-1}) &= \phi_{j/m}(\mu w), \\
&\cdots \\
q_0'(\phi_0)\phi_m(w) + F_m(\phi_0, \cdots, \phi_{m-1}) &= \phi_1(\mu w). \\
&\cdots
\end{aligned}
$$

$$\phi_1(w) = -\frac{q_1(\phi_0(w))}{q_0'(\phi_0(w))},$$

$$\cdots$$

$$\phi_m(w) = \frac{\phi_1(\mu w)}{q_0'(\phi_0(w))} - \frac{F_m(\phi_0, \cdots, \phi_{m-1})}{q_0'(\phi_0(w))}$$

$$= -\frac{q_1(\phi_0(\mu w))}{q_0'(\phi_0(w))q_0'(\phi_0(\mu w))} - \frac{F_m(\phi_0, \cdots, \phi_{m-1})}{q_0'(\phi_0(w))},$$

$$\cdots .$$

$$\phi_{m^k}(w) = -\frac{q_1(\phi_0(\mu^k w))}{\prod_{j=0}^{k} q_0'(\phi_0(\mu^j w))} - \frac{F_{m^k}(\phi_0, \phi_1, \cdots, \phi_{m^k-1})}{\prod_{j=0}^{k-1} q_0'(\phi_0(\mu^j w))}.$$

By the assumptions, it follows that,

for any k, ϕ_{m^k} has a pole at $w = \dfrac{w_0}{\mu^k} \to 0$.

Thus ϕ cannot be holomorphic at the origin.

Convergence of formal conjugacies

$$f(z,w) = \left(z^m, (w+1)^d - 1 + \sum_{j=0}^{d} \sum_{i \geq 1} q_{i,j} z^i w^j\right).$$

Theorem 5. *Assume $d \leq m$. Then, there exists a holomorphic conjugacy Φ of f to $f_0(z,w) = (z^m, dw)$ at the origin.*

As a corollary, Theorems 2 and 3 hold for

$$f(z,w) = (z^2, w^2 + 2(1 - bz)w),$$

with appropriate b.

In case $\alpha_- = (\beta, 0)$ is a super-saddle: $\mu_- = q'_\beta(0) = 0$

Put $q_\beta(w) = h(w)w^m$, $h(0) \neq 0$, $m \geq 2$. Then

$$q_z(w) = h(w)w^m + \sum_{i,j\geq 1, i+j\leq d} b_{i,j}(z-\beta)^i w^j.$$

Lemma 2. *Suppose $b_{1,1} \neq 0$. Then*

$$\exists \text{ a critical point } c(z) \text{ of } q_z \text{ s.t. } c(z) \asymp (z-\beta)^{1/(m-1)},$$

$$\exists c_k(z) \in (Q_z^k)^{-1}(c(p^k(z))) \text{ s.t. } c_k(z) \asymp (z-\beta)^{1/m^k(m-1)},$$

for any $k \geq 1$ and for z close to β.

Proof. The root $c(z)$ of the equation :

$$q_z'(w) = \tilde{h}(w)w^{m-1} + b_{1,1}(z-\beta)(1+O(z-\beta,w)) = 0,$$

with the property $c(\beta) = 0$ satisfies $c(z) \asymp (z-\beta)^{1/(m-1)}$.

The estimate for $c_k(z)$ follows from :

$$\begin{aligned} Q_z^k(w) &= Q_\beta^k(w) + H(z,w)(z-\beta)w \\ &= h_k(w)w^{m^k} + H(z,w)(z-\beta)w, \\ c(p^k(z)) &\asymp (p^k(z)-\beta)^{1/(m-1)} \asymp (z-\beta)^{1/(m-1)}. \end{aligned}$$

□

Corollary 1. *If* $b_{1,1} \neq 0$, f *is not rigid at* $(\beta, 0)$.

Take a sequence $z_n \in U_0$ tending to β.

Theorem 6. *Suppose $b_{1,1} \neq 0$. For any sequence $k_n \to \infty$, if $Q_{z_n}^{k_n} \to g$, then $g \equiv 0$.*

Proof. If $Q_{z_n}^{k_n} \to g$, then $(Q_{z_n}^{k_n})' \to g'$.

By Lemma 2, arbitrarily many zeros $\{c_k(z_n)\}_{k<k_n}$ of $(Q_{z_n}^{k_n})'$ merge into 0.

Thus $g' \equiv 0$, hence $g(w) \equiv g(0) = 0$. $\square$

It also happens that

$$J_z \to J(g) = \overline{g^{-1}(J_0)} = \overline{W^s(\alpha_-)} \quad \text{as} \quad z \to \beta.$$

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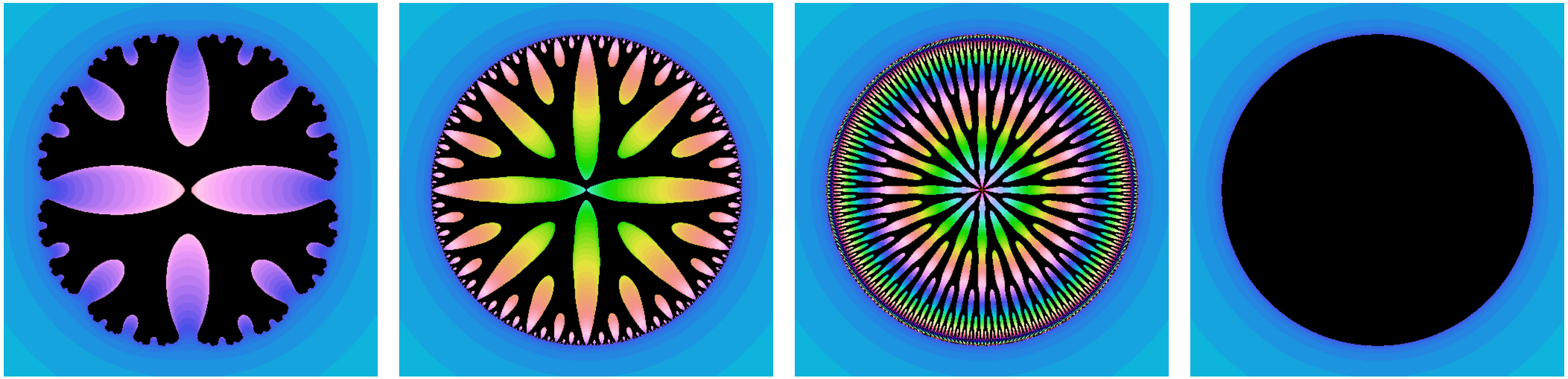

From left: $z = 0.88,\ 0.899,\ 0.899999999,\ 0.9\,(= \beta)$

This is the case : $J_z \to K_\beta = \overline{W^s(\alpha_-)}$.

$$f(z,w) = (z^3, w^3 - \frac{3i}{\sqrt{2}} z w^2 + 3(1-z)(w+1)w).$$

$q_0(w) = (w+1)^3 - 1,$

$q_1(w) = w^3 - \frac{3i}{\sqrt{2}} w^2$ has two super-attracting fixed points 0 and $\sqrt{2}i$.

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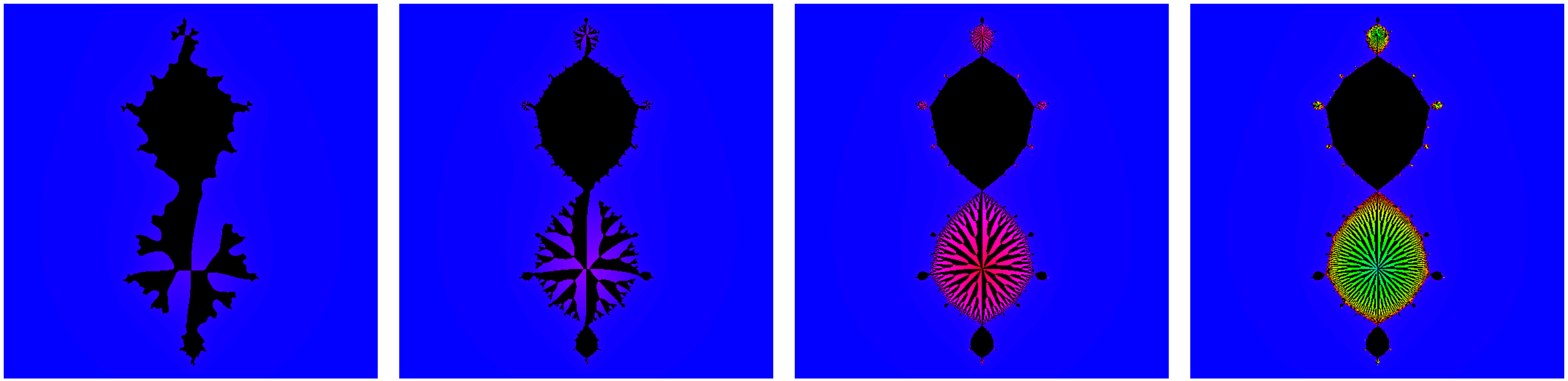

From left: $z = 0.98,\ 0.999,\ 0.99999,\ 0.9999999999$