Discontinuity of the escape rate of a degenerating meromorphic family of rational maps

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A degenerating meromorphic family of rational functions

Fix an integer $d > 1$, and

- $f : \mathbb{D}^* \times \mathbb{P}^1 \to \mathbb{P}$, or informally $(f_t)_{t\in\mathbb{D}^*}$, is a holomorphic family of rational functions of degree d parametrized by $\mathbb{D}^*$, but more specifically,
- suppose that $f_t(z) \in \mathcal{O}(\mathbb{D})[t^{-1}](z)$, i.e.,

1.
$$f_t(z) = \frac{\sum_{j=0}^{d} b_j(t) z^j}{\sum_{k=0}^{d} a_k(t) z^k}, \quad \exists a_k(t), \exists b_j(t) \in \mathcal{O}(\mathbb{D}),$$
(multiplying both denominator and numerator in $\mathcal{O}(\mathbb{D})[t^{-1}]^*$ is ambiguity of representation),
2. and that for each $t \in \mathbb{D}^*$, f_t is a rational function$/\mathbb{C}$ on $\mathbb{P}^1$ of degree d.

Then $\exists$a finite subset $H \subset \mathbb{P}^1$, $\exists \phi \in \mathbb{C}(z)$ of $\deg \phi \in \{0, 1, \ldots, d\}$,

$$\lim_{t\to 0} f_t = \phi \quad \text{on } \mathbb{P}^1 \setminus H$$

locally uniformly. For simplicity, we say ϕ is a degenerating limit of the family f.

A (holomorphically) marked point in $\mathbb{P}^1$

In addition to the family f,

- a is a marked point in $\mathbb{P}^1$ (holomorphically) parametrized by $\mathbb{D}^*$, but more specifically,
- suppose that $a(t) \in \mathcal{O}(\mathbb{D})[t^{-1}]$, so we can write as

$$a(t) = \frac{\tilde{a}_1(t)}{\tilde{a}_0(t)}, \quad \exists \tilde{a}_0(t), \exists \tilde{a}_1(t) \in \mathcal{O}(\mathbb{D})$$

(again, multiplying both denominator and numerator in $\mathcal{O}(\mathbb{D})[t^{-1}]^*$ is ambiguity of representation).

The escaping rate function and the activity current associated to (f, a)

Taking the family of homogeneous polynomial maps

$$\tilde{f}_t(z_0, z_1) := \Big(\sum_{k=0}^{d} a_k(t) z_0^{d-k} z_1^k, \sum_{j=0}^{d} b_j(t) z_0^{d-j} z_1^j\Big) : \mathbb{C}^2 \to \mathbb{C}^2$$

(a lift of f), there is the locally uniform limit

$$G_{\tilde{f}}(t, p) := \lim_{n \to \infty} \frac{1}{d^n} \log \|\tilde{f}_t^n(p)\| \quad \text{on } \mathbb{D}^* \times (\mathbb{C}^2 \setminus \{0\})$$

($\|\cdot\|$: any norm on $\mathbb{C}^2$). Taking also the holomorphic map

$$\tilde{a}(t) := (\tilde{a}_0(t), \tilde{a}_1(t)) : \mathbb{D}^* \to \mathbb{C}^2 \setminus \{0\}$$

(a lift of a), we obtain the escape rate function

$$t \mapsto G_{\tilde{f}}(t, \tilde{a}(t)) \quad \text{on } \mathbb{D}^*$$

associated to $(\tilde{f}, \tilde{a})$, which is a <u>continuous and subharmonic</u> function on $\mathbb{D}^*$.

Theorem 1.1 (DeMarco)

For every $\tilde{f}, \tilde{a}$ *as the above, there is a constant* $\alpha = \alpha_{\tilde{f},\tilde{a}}$ *such that*

$$g_{f,a}(t) := G_{\tilde{f}}(t, \tilde{a}(t)) + \alpha \cdot \log(|t|^{-1}) = o(\log|t|) \quad \text{as } t \to 0. \quad (*)$$

(*ambiguity of* $g_{f,a}$ *is exactly to add a function harmonic on* $\mathbb{D}^*$ *and bounded around* $t = 0$.)

The activity measure associated to (f, a) is

$$\mu_{f,a} := \mathrm{dd}^c_t G_{\tilde{f}}(t, \tilde{a}(t)) (= \mathrm{dd}^c g_{f,a}) \quad \text{on } \mathbb{D}^*$$

($\mathrm{supp}\,\mu_{f,a} =$ the activity locus associated to (f, a) in McMullen's sense).

We are interested in (the rationality of α and) the continuous extendability of $g_{f,a}$ across $t = 0$.

A special case: the Lyapunov exponent function associated to f

Taking a finite and possibly ramified holomorphic self-covering π of the parameter space $\mathbb{D}^*$ if necessary, there are $2d-2$ marked points $c_1, \ldots, c_{2d-2} \in \mathcal{O}(\mathbb{D})[s^{-1}]$ such that for $\forall s \in \mathbb{D}^*$, $c_1(s), \ldots, c_{2d-2}(s)$ are all the critical points of $f_{\pi(s)}$, taking into account their multiplicities.

DeMarco defined the bifurcation measure (current) associated to f as

$$T_f := \pi_* \left(\sum_{j=1}^{2d-2} \mu_{f_{\pi(\cdot)}, c_j} \right) \quad \text{on } \mathbb{D}^*,$$

which is independent of the choice of the covering π.

The Lyapunov exponent function

$$t \mapsto L(f_t) := \int_{\mathbb{P}^1} \log |Df_t|_{T\mathbb{P}^1} \mathrm{d}\mu_{f_t} \quad \text{on } \mathbb{D}^*$$

(for each $t \in \mathbb{C}^*$, μ_{f_t} is the unique maximal entropy measure of f_t on $\mathbb{P}^1$) is a potential of the bifurcation measure T_f: indeed and moreover,

$$L(f_t) = -\log d + \sum_{j=1}^{2d-2} G_{\tilde{f}}(t, \tilde{c}_j(t)) - \frac{2}{d} \log \left| \mathrm{Res}\left(\tilde{f}\right)\right| \quad \text{(DeMarco's formula)}$$

$$= \eta \log(|t|^{-1}) + o(\log |t|) \quad \text{as } t \to 0 \qquad (**)$$

for some constant $\eta \geq 0$ ($\det DF = \prod_{j=1}^{2d-2} (\cdot \wedge \tilde{c}_j)$).

Again, we are interested in (the rationality of $\eta = \eta_f$ and) the continuous extendability of the function $t \mapsto L(f_t) - \eta \log(|t|^{-1})$ across $t = 0$.

An example of the continuous extendability

Set $M :=$ (the Mandelbrot set) and

$$g_{M,\infty}(t) := \text{the Green function of } M \text{ with pole } \infty$$
$$= \log|t| + (\exists \text{harmonic function on } \mathbb{P}^1 \setminus \{|z| \le \exists R\}) \text{ around } \infty.$$

When $f_t(z) = z^2 + t^{-1}$ and $c(t) \equiv 0$ $(t \in \mathbb{D}^*)$, then we have

$$g_{f,c}(t) = \frac{1}{2} g_{M,\infty}(t^{-1}) - \frac{1}{2}\log|t|^{-1} \quad \text{on } \mathbb{D}^*,$$

so $g_{f,c}$ extends harmonically, so continuously, around $t = 0$.

Also, by the Manning-Przytycki formula,

$$L(f_t) = \log d + \frac{1}{2} g_{M,\infty}(t^{-1}) \quad \text{on } \mathbb{D}^*,$$

so that $\eta = \frac{1}{2}$ and that $L(f_t) - \frac{1}{2}\log|t|^{-1} = \log d + g_{f,c}(t)$ also extends harmonically, so continuously, around $t = 0$.

The continuity of $g_{f,a}$ across $t=0$

DeMarco also wrote $g_{f,a}$ as

$$g_{f,a}(t) = G_{\tilde{f}}(t,\tilde{a}(t)) + \exists \left(\lim_{n\to\infty} \frac{\min_{j\in\{0,1\}} \mathrm{ord}_{t=0}(\tilde{f}_t^n)_j(\tilde{a}(t))}{d^n}\right) \log|t|^{-1}$$

$$= \lim_{n\to\infty} \frac{1}{d^n} \left(\log\|\tilde{f}_t^n(\tilde{a})\| + \left(\min_{j\in\{0,1\}} \mathrm{ord}_{t=0}(\tilde{f}_t^n)_j(\tilde{a}(t))\right) \log|t|^{-1} \right)$$

on $\mathbb{D}^*$, where $\mathrm{ord}_{t=0}$ is the zeros order at $t=0$, and $\tilde{f}_t^n =: ((\tilde{f}_t^n)_0, (\tilde{f}_t^n)_1)$ for $\forall n \in \mathbb{N}$, and asked when the latter convergence can be locally uniform on $\mathbb{D}$.

Correspondingly, Charles Favre conjectured that the function $t \mapsto L(f_t) - \eta_f \log|t|^{-1}$ would extend across $t=0$ (even in higher dimension).

For families of polynomials, the answer is YES:

Theorem 2.1 (Favre–Gauthier)

When $f \in \mathcal{O}(\mathbb{D})[t^{-1}][z]$ and $a \in \mathcal{O}(\mathbb{D})$, $g_{f,a}$ extends continuously to $\mathbb{D}$.

Well ..., for $f \in \mathcal{O}(\mathbb{D})[t^{-1}](z)$ (and $a \in \mathcal{O}(\mathbb{D})[t^{-1}]$), the situation is different.

Cooking a discontinuous $g_{f,a}$

Prepare

- a rational function $\phi(z) \in \mathbb{C}(z)$ of degree > 0, and
- a point $a_0 \in \mathbb{C}$ neither periodic nor preperiodic under ϕ and satisfying

$$\left(\bigcap_{N \in \mathbb{N}} \overline{\{\phi^n(a_0) : n \geq N\}} \right) \cap \{\phi^n(a_0) : n \in \mathbb{N} \cup \{0\}\} \neq \emptyset.$$

Find

- a point $h \in \left(\bigcap_{N \in \mathbb{N}} \overline{\{\phi^n(a_0) : n \geq N\}} \right) \setminus \{\phi^n(a_0) : n \in \mathbb{N} \cup \{0\}\}$ and
- a sequence (n_j) in $\mathbb{N}$ tending to ∞ as $j \to \infty$

such that the sequence $(\phi^{n_j}(a_0))_j$ tends to h very quickly as $j \to \infty$.

Pick

- $\epsilon > 0$ so that ϕ has neither zeros nor poles in $\{z : 0 < |z - h| < \epsilon\}$ and
- $m \in \mathbb{N}$.

(the ingredients have 6 items)

Define

$$f_t(z) := \phi(z) \cdot \frac{z - h + \epsilon t^m}{z - h - \epsilon t^m} \in \mathcal{O}(\mathbb{D})[t^{-1}](z) \quad \text{of degree } \deg \phi + 1,$$

and pick $\forall a \in \mathcal{O}(\mathbb{D})$ satisfying $a(0) = a_0$.
Then $\exists C = C_{f,a} \in \mathbb{R}$ such that

$$\limsup_{t \to 0} g_{f,a}(t) \leq C + \frac{1}{2} \cdot \frac{\log[\phi^{n_j}(a_0), h]}{(\deg \phi + 1)^{n_j+1}} \quad \text{for } \forall j \in \mathbb{N}$$

($[z, w]$: the chordal metric on $\mathbb{P}^1$ normalized as $[0, \infty] = 1$).
We can always choose h and (n_j) so that

$$\lim_{j \to \infty} \frac{\log[\phi^{n_j}(a_0), h]}{(\deg \phi + 1)^{n_j+1}} = -\infty.$$

Consequently, using this recipe, we cooked up

Theorem 1 (DeMarco-Ok)

There are f and a such that $\lim_{t \to 0} g_{f,a}(t) = -\infty$.

A few applications of our recipe

1. Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$, set $\phi(z) = e^{2i\pi\theta} z$ and $a_0 = 1$, and pick $h, (n_j), \epsilon$, and $m = 1$ as the above.
   The obtained family $f \in \mathcal{O}(\mathbb{D})[t^{-1}](z)$ is of degree $1 + 1 = 2$, and for any marked point $a \in \mathcal{O}(\mathbb{D})$ satisfying $a(0) = a_0 = 1$, we have $\lim_{t\to 0} g_{f,a}(t) = -\infty$.
2. Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and an integer $d> 2$, and set

$$\phi(z) = e^{2i\pi\theta}(z - a_0)^{d-1} \quad \text{and} \quad a_0 = \frac{e^{2i\pi\theta} - (d-1)}{(d-1)^{(d-1)/(d-2)}}.$$

   Then ϕ has an irrationally indifferent fixed point having the multiplier $e^{2i\pi\theta}$ and the unique critical point a_0 in $\mathbb{C}$, so satisfy the assumption (by Mañé's theorem), and pick $h, (n_j), \epsilon$, and $m = 1$ as the above.
   The obtained family $f \in \mathcal{O}(\mathbb{D})[t^{-1}](z)$ is of degree $(d-1) + 1 = d > 2$, and the constant mapping $c(t) \equiv a_0$ is a marked critical point of f. We have $\lim_{t\to 0} g_{f,c}(t) = -\infty$.

3 Fix $\theta \in \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} \{\theta \in \mathbb{R} : [1, e^{2in\pi\theta}] < e^{-(n-1)2^n}\} \setminus \mathbb{Q}$, and set

$$\phi(z) = e^{2i\pi\theta} z,\ h = 1,\ \epsilon = 1,\ \text{and } f_t(z) = \phi(z) \cdot \frac{z - h - \epsilon t^2}{z - h + \epsilon t^2}.$$

This family f is of degree $1 + 1 = 2$, and has two marked critical points $c_{\pm} \in \mathcal{O}(\mathbb{D})$ (and $c_{\pm}(0) = 1 = h$) and two marked critical values $v_{\pm}(t) := f_t(c_{\pm}(t)) \in \mathcal{O}(\mathbb{D})$, and in fact $v_{\pm}(0) = \phi(c_{\pm}(0)) = e^{2i\pi\theta}$.

We can fix (n_j) in $\mathbb{N}$ tending to ∞ as $j \to \infty$ such that for $\forall j \in \mathbb{N}$,

$$[\phi^{n_j-1}(e^{2i\pi\theta}), h] = [1, e^{2in_j\pi\theta}] < e^{-(n_j-1)2^{n_j}}.$$

Applying our recipe to ϕ, $a_0 := e^{2i\pi\theta}$, h, and (n_j), we have

$$\lim_{t\to 0} g_{f,v_{\pm}}(t) = -\infty,$$

and on the other hand, we also have

$$g_{f,c_{\pm}} = \frac{1}{2} g_{f,v_{\pm}}(t) \quad \text{on } \mathbb{D}^*.$$

Hence $\lim_{t\to 0} g_{f,c_{\pm}}(t) = -\infty$.

Recall that, by the DeMarco formula,

$$L(f_t) - \eta_f \log|t|^{-1} = \sum_{j=1}^{2d-2} g_{f,c_j}(t) \quad \text{on } \mathbb{D}^*,$$

and by (*), $g_{f,a}$ always extends subharmonically to $\mathbb{D}$, so bounded from above around $t = 0$.

4. Hence in the 2nd and the 3rd items, for the families f made by our recipe, we also have

$$\lim_{t \to 0} (L(f_t) - \eta_f \log|t|^{-1}) = -\infty.$$

Remark 2.1

The proof of Theorem 1 (finding the discontinuity of $g_{f,a}$) is based on DeMarco's iteration formula applied to the *degenerate* homogeneous polynomial lift $\tilde{f}_0 : \mathbb{C}^2 \to \mathbb{C}^2$ (under the assumption that $\deg \phi > 0$) and on the Baire category theorem for finding h from a_0 satisfying the assumption.

1. The potential $g_{f,a}$ of the activity measure $\mu_{f,a}$ on $\mathbb{D}^*$

2. The discontinuity of $g_{f,a}$ across the puncture $t = 0$ (with Laura DeMarco)

3. References

References and Thanks

For more details on the (dis)continuity problem in this talk,

- LAURA DEMARCO, Bifurcations, intersections, and heights, Algebra and Number Theory, 10(5):1031-1056 (2016).
- CHARLES FAVRE, Degeneration of endomorphisms of the complex projective space in the hybrid space, *ArXiv e-prints* (November 2016).
- CHARLES FAVRE AND THOMAS GAUTHIER, Continuity of the Green function in meromorphic families of polynomials, *ArXiv e-prints* (June 2017).
- LAURA DEMARCO AND YÛSUKE OKUYAMA, Discontinuity of a degenerating escape rate, *ArXiv e-prints* (October 2017).

For the bifurcation and activity currents associated to a holomorphic family and a marked point, please see DeMarco (PhD thesis, 2001, 2003) and Dujardin–Favre (2008). Berteloot's survey is also useful.
For a degenerating meromorphic family, see DeMarco (2005).

どうも有り難うございました。

Thank you very much for paying your attention!