Escaping singular orbits in Class $\mathcal{B}$

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Notable sets

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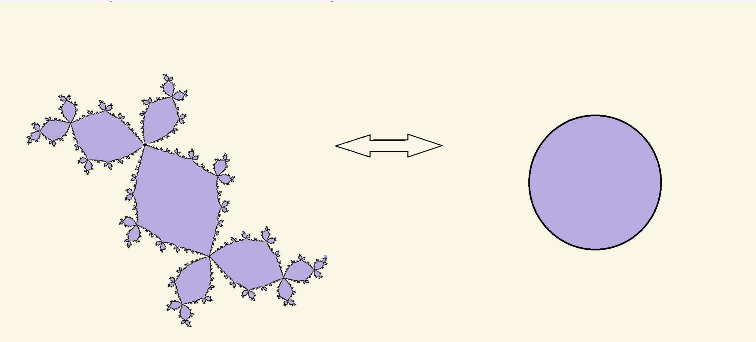
- The **escaping set** consists of the points that escape to infinity under iteration:

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}$$

Frequently structured in curves called **dynamic rays.**

Structure of the escaping set

- In the *polynomial* case: when $J(f)$ is connected, rays as preimages of radial lines under Böttcher's map.



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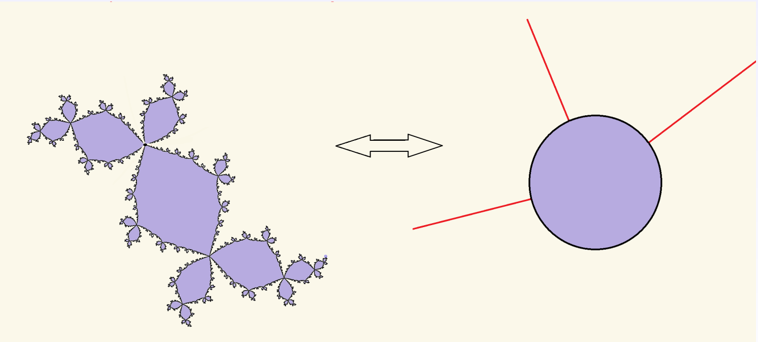

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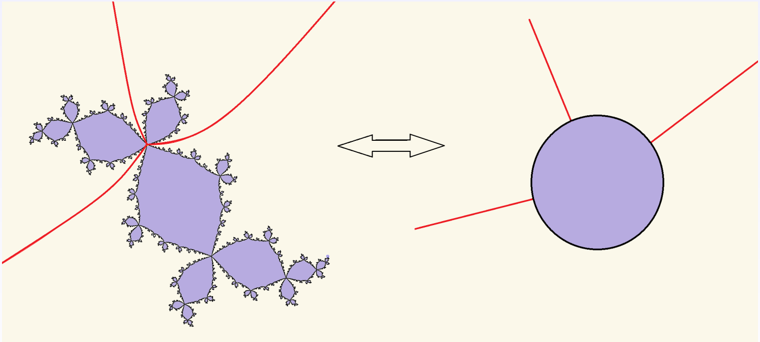

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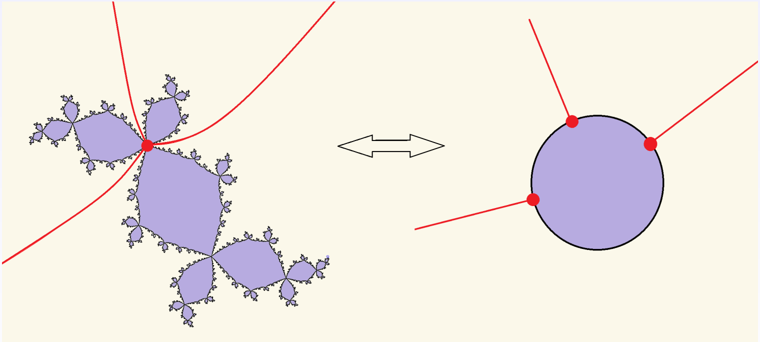

All rays **land** if and only if $J(f)$ is **locally connected.**

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  - True for functions of finite order in class $\mathcal{B}$.([RRRS]).

Singular values

The set of **singular values** $S(f)$ is the smallest closed subset of $\mathbb{C}$ such that $f : \mathbb{C} \setminus S(f) \to \mathbb{C} \setminus S(f)$ is a covering map.

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The **postsingular set** of f is defined as

$$P(f) = \overline{\bigcup_{n \geq 0} f^n(S(f))}.$$

- ⋆ $f^k : \mathbb{C} \setminus \mathcal{O}^-(S(f)) \to \mathbb{C} \setminus P(f)$ is a covering map for all $k \geq 0$.

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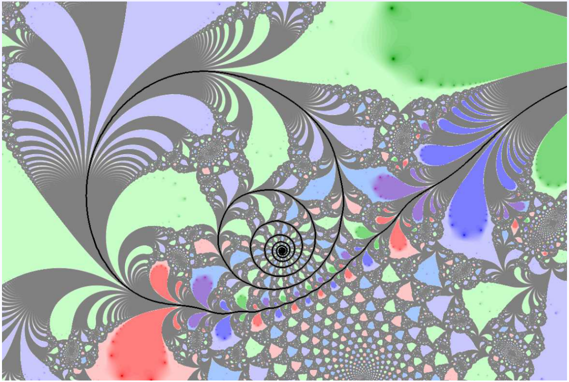
- We say that γ **lands** at z if $\lim_{t\to 0} \gamma(t) = z$.

Landing rays for transcendental functions

- In the **exponential family**, $E_\lambda(z) = \lambda e^z$, all rays land when E_λ has an attracting or parabolic orbit. (Devaney '93, Devaney & Jarque '01, Rempe-Gillen '06).



*Picture by Rempe-Gillen

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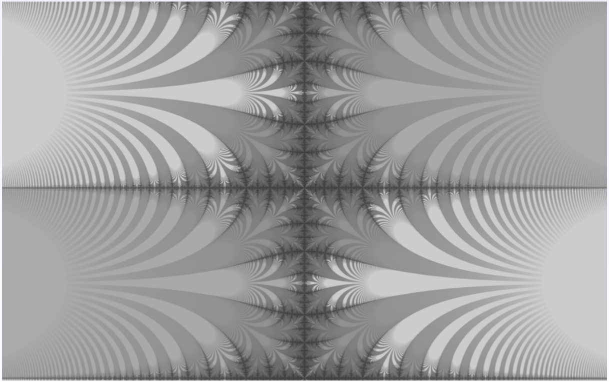
- In the **cosine family**, $E_{a,b}(z) = ae^z + be^{-z}$, all rays land when $P(f)$ is strictly preperiodic (Schleicher '06) .



*Picture by Arnaud Chritat

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If f is of finite order and strongly subhyperbolic, then every dynamic ray of $J(f)$ lands

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If f is of finite order and strongly subhyperbolic, then every dynamic ray of $J(f)$ lands and every point in $J(f)$ is a landing point or in a dynamic ray.

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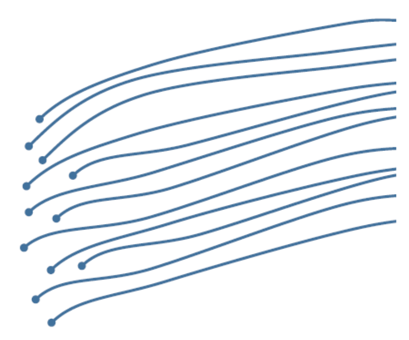

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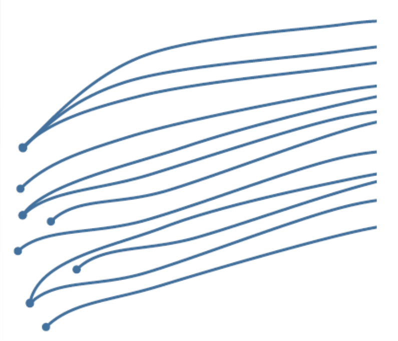

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Theorem ([Rempe-Gillen ('09)])

There exist a constant $R > 0$ and a quasiconformal map $\vartheta : \mathbb{C} \to \mathbb{C}$ such that $\vartheta \circ f = g_\lambda \circ \vartheta$ for all $z \in J_R(g)$, with

$$J_R(g_\lambda) := \{z \in J(g) : |g_\lambda^n(z)| \geq R \text{ for all } n \geq 1\}.$$

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Strategy: Extend the conjugacy near infinity to $J(f)$ by considering the model space

$$(J(g)_{\pm}, \tau),$$

with $J(g)_{\pm} := J(g) \times \{-, +\}$ and τ appropriate topology that preserves the circular order of the rays, using the map $\tilde{g} : J(g)_{\pm} \to J(f)$

$$\tilde{g}(z, \sigma) := (g(z), \sigma).$$

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There exists a continuous surjective function $\varphi : \mathcal{J}(g)_\pm \to \mathcal{J}(f)$ so that

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Moreover, $\varphi(I(g)_\pm) = I(f)$.

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Thanks for your attention!