Multiply connected wandering domains and commuting functions

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Basic definitions

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The **Julia set** (or chaotic set) is

$$J(f) = \mathbb{C} \setminus F(f).$$

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The **fast escaping set** is $A(f) = \bigcup_{L \in \mathbb{N}} f^{-L}(A_R(f))$ where:

$$A_R(f) = \{z \in \mathbb{C} : |f^n(z)| \geq M^n(R)\ \forall\ n \in \mathbb{N}\},$$

if $R > 0$ is such that $M^n(R) \to \infty$ as $n \to \infty$.

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3. **Siegel disc** There is a conformal mapping $\phi : U \to \mathbb{D}$, where $\mathbb{D}$ is the unit disc, such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\theta} z$, where θ is irrational.

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4. **Baker domain** For all $z \in U$, we have $f^{np}(z) \to \infty$ as $n \to \infty$.

Wandering domains

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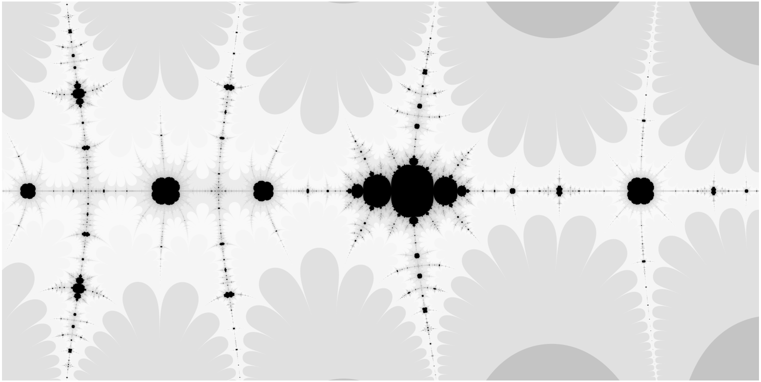
Question Can we produce a classification of wandering domains?

A wandering cauliflower



$$f(z) = z\cos z + 2\pi$$

Basic properties of multiply connected Fatou components

Baker (for first two), Rippon and Stallard

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- For large n, each component of the boundary of U_n is a component of $A_R(f) \cap J(f)$.
- $A_R(f)$ is a spider's web

Structure of multiply connected wandering domains

Bergweiler, Rippon and Stallard

Theorem

If U is a multiply connected wandering domain and $D \subset U$ is an open neighborhood of z_0, then there exists $\alpha > 0$ such that, for large $n \in \mathbb{N}$,

$$f^n(D) \supset A(|f^n(z_0)|^{1-\alpha}, |f^n(z_0)|^{1+\alpha}).$$

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such that, for every compact set $C \subset U$,
$f^n(C) \subset B_n$ for $n \geq N(C)$.

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Connectivity of multiply connected wandering domains
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- *the inner connectivity of U_n is finite and decreases to 2.*

The **outer connectivity** of U_n is defined in a similar way but can decrease from infinity to a finite number.

Connectivity of multiply connected wandering domains

discussed with Baumgartner and Bergweiler

Theorem

Let U be a multiply connected wandering domain.

- *If U has infinite inner connectivity then U has uncountably many complementary components (only countably many of which can have interior) and hence $A_R(f) \cap J(f)$ has uncountably many components.*

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- *If U has finite inner connectivity then U has only countably many complementary components.*

Corollary

There are examples of functions for which $A_R(f) \cap J(f)$ has only countably many components.

Applications to a question about commuting functions

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If f, g are analytic with $f \circ g = g \circ f$, does $J(f) = J(g)$?

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Fatou showed that $g(F(f)) \subset F(f)$ and hence $F(f) \subset F(g)$ by Montel's Theorem.

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Julia's proof

Based on repelling periodic points (in $J(f)$!).

Transcendental functions

Theorem (Baker, 1962)

If $f \circ g = g \circ f$ and U is a Fatou component of f, then $g(U) \subset F(f)$ unless $U \subset I(f)$.

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Corollary

If $f \circ g = g \circ f$ and f, g have no fast escaping Fatou components, then $J(f) = J(g)$.

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Theorem (Rippon and Stallard, 2005)

If U is a multiply connected wandering domain, then U is fast escaping.

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Theorem (Rippon and Stallard, 2005)

If U is a multiply connected wandering domain, then U is fast escaping.

There are only two known examples of functions with simply connected fast escaping wandering domains:
one due to Bergweiler and one due to Sixsmith.

Multiply connected wandering domains

Using the description of multiply connected wandering domains given by Bergweiler, Rippon and Stallard, we were able to prove the following.

Theorem (Benini, Rippon and Stallard, 2015)

If $f \circ g = g \circ f$ and U is a multiply connected wandering domain of f, then $g(U) \subset F(f)$.

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If $f \circ g = g \circ f$ and U is a multiply connected wandering domain of f, then $g(U) \subset F(f)$.

Corollary

If $f \circ g = g \circ f$ and f, g have no simply connected fast escaping wandering domains, then $J(f) = J(g)$.