Weak Mean Stability in Random Holomorphic Dynamical Systems

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Definition 1.

(1) Let $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \cong S^2$ be the Riemann sphere endowed with the spherical distance d.

(2) Let Rat $:= \{f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid f$ is non-constant and holomorphic$\}$ endowed with the distance η, where $\eta(f,g) = \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$. Note that (Rat, η) is a complete separable metric space.

(3) For a metric space Y, we denote by $\mathfrak{M}_1(Y)$ the space of all Borel probability measures on Y.

(4) For a subset Y of Rat, we set
$\mathfrak{M}_{1,c}(Y) := \{\tau \in \mathfrak{M}_1(Y) \mid \text{ supp}\,\tau \text{ is a compact subset of } Y\}$.

(5) For a $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$, we set
$G_\tau := \{\gamma_n \circ \cdots \circ \gamma_1 \mid n \in \mathbb{N}, \gamma_j \in \text{supp}\,\tau (\forall j)\}$. Note that this is a semigroup whose product is the composition of maps.

(6) We say that an element $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ is weakly mean stable if there exist an $n \in \mathbb{N}$, an $m \in \mathbb{N}$, non-empty open subsets $U_1, \ldots, U_m$ of $\hat{\mathbb{C}}$, a non-empty compact subset K of $\hat{\mathbb{C}}$ with $K \subset \cup_{j=1}^{m} U_j$, and a constant c with $0 < c < 1$ such that the following (a) (b) (c) hold.

(a) For each $(\gamma_1, \ldots, \gamma_n) \in (\text{supp}\, \tau)^n$, we have

$$\gamma_n \circ \cdots \circ \gamma_1(\cup_{j=1}^{m} U_j) \subset K.$$

Moreover, for each $j = 1, \ldots, m$, for all $x, y \in U_j$ and for each $(\gamma_1, \ldots, \gamma_n) \in (\text{supp}\, \tau)^n$, we have

$$d(\gamma_n \circ \cdots \circ \gamma_1(x), \gamma_n \circ \cdots \circ \gamma_1(y)) \leq c d(x, y).$$

(b) Let $D_\tau := \bigcap_{h \in G_\tau} h^{-1}(\hat{\mathbb{C}} \setminus \cup_{j=1}^{m} U_j)$. Then $\sharp D_\tau < \infty$.

(c) For each minimal set L of τ with $L \subset D_\tau$, there exist a $z \in L$ and an $\alpha \in G_\tau$ such that $\alpha(z) = z$ and $|\alpha'(z)| > 1$.
Here, a non-empty compact subset L of $\hat{\mathbb{C}}$ is said to be a minimal set of τ if for each $z \in L$, $\overline{\cup_{h \in G_\tau} \{h(z)\}} = L$.

(7) For each $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$, we define $M_\tau^* : \mathfrak{M}_1(\hat{\mathbb{C}}) \to \mathfrak{M}_1(\hat{\mathbb{C}})$ as follows.

$$M_\tau^*(\mu)(A) := \int \mu(h^{-1}(A))\ d\tau(h)$$

for each $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}})$ and for each Borel subset A of $\hat{\mathbb{C}}$.

Theorem 2 ([4]). *Let $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ be weakly mean stable.*
Then there exists an $l \in \mathbb{N}$ such that for each $x \in \hat{\mathbb{C}}$, there exists an $(M_\tau^*)^l$-invariant $\mu_x \in \mathfrak{M}_1(\hat{\mathbb{C}})$ such that

$$(M_\tau^*)^{nl}(\delta_x) \to \mu_x \quad \textit{as } n \to \infty$$

in $\mathfrak{M}_1(\hat{\mathbb{C}})$ with respect to the weak convergence topology.
Here, δ_x denotes the Dirac measure at x.

Theorem 3 ([4])**.** *Let* $\tau \in \mathfrak{M}_{1,c}(Rat)$ *be weakly mean stable. Let*

$J(G_\tau) := \{z \in \hat{\mathbb{C}} \mid$ *for any nbd* U *of* z *in* $\hat{\mathbb{C}}, G_\tau$ *is not equicontinuous on* $U\}$.

Suppose we have the following (1) *and* (2).

(1) $\sharp J(G_\tau) \geq 3$. (Note*: if* $\exists g \in$ *supp* τ *with* $\deg(g) \geq 2$*, then* $\sharp J(G_\tau) \geq 3$.)

(2) *For each minimal set* L *of* τ *with* $L \subset D_\tau$*, where* D_τ *is the set coming from Definition 1 (6), we have the following* (a)(b).

  (a) *The Lyapunov exponent* $\chi(L, \tau)$ *of* (L, τ) *is not zero.*

  (b) *If* $\chi(L, \tau) > 0$*, then for each* $z \in L$ *and for each* $h \in$ *supp* τ*, we have* $Dh_z \neq 0$.

Then, there exist a subset Ω_τ of $\hat{\mathbb{C}}$ with $\sharp(\hat{\mathbb{C}} \setminus \Omega_\tau) \leq \aleph_0$ and a constant c_τ with $c_\tau < 0$ such that the following holds.

- *For each $z \in \Omega_\tau$, there exists a Borel subset $B_{\tau,z}$ of $(\mathrm{Rat})^{\mathbb{N}}$ with $(\otimes_{n=1}^{\infty}\tau)(B_{\tau,z}) = 1$ such that for each $(\gamma_1, \gamma_2 \ldots,) \in B_{\tau,z}$, we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|D(\gamma_n \circ \cdots \gamma_1)_z\| \leq c_\tau < 0.$$

Remark 4. Statements of Theorems 2 and 3 cannot hold for deterministic dynamics of a single $f \in \mathrm{Rat}$ with $\deg(f) \geq 2$.
In fact, in the Julia set $J(f)$ of f, we have a chaotic phenomenon.
See Mañé's paper (1988)[1] etc.

Theorem 5 ([4]). *Let Y be one of the following (1)–(4).*

(1) $\{f \in \text{Rat} \mid f$ *is a polynomial with* $\deg(f) \geq 2\}$.

(2) $\{\lambda z(1-z) \in \text{Rat} \mid \lambda \in \mathbb{C} \setminus \{0\}\}$.

(3) $\{z - \lambda \frac{f(z)}{f'(z)} \in \text{Rat} \mid \lambda \in \mathbb{C}, |\lambda - 1| < 1\}$ *where f is a polynomial with* $\deg(f) \geq 2$. *Note that this family is related to "random relaxed Newton's methods for f" in which we can find roots of any polynomial f more easily than deterministic Newton's method* ([4]).

(4) $\{z + \lambda f(z) \in \text{Rat} \mid \lambda \in \mathbb{C} \setminus \{0\}\}$ *where f is a polynomial with* $\deg(f) \geq 2$ *such that for each $z_0 \in \mathbb{C}$ with $f(z_0) = 0$, we have* $f'(z_0) \neq 0$.

Then there exists an open and dense subset A of $\mathfrak{M}_{1,c}(Y)$ such that each $\tau \in A$ is weakly mean stable and satisfies the assumptions of Theorems 2 and 3 (thus the statements of Theorems 2 and 3 hold for τ).

Here, we endow $\mathfrak{M}_{1,c}(Y)$ with the topology such that a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ in $\mathfrak{M}_{1,c}(Y)$ tends to an element $\tau \in \mathfrak{M}_{1,c}(Y)$ if and only if

(a) *for each bounded continuous function $\varphi : Y \to \mathbb{R}$, we have $\int_Y \varphi d\tau_n \to \int_Y \varphi d\tau$ as $n \to \infty$, and*

(b) *supp $\tau_n \to$ supp τ as $n \to \infty$ with respect to the Hausdorff metric in the space of all non-empty compact subsets of Y.*

Theorem 6 ([4]). (Random relaxed Newton's methods)
Let f be a polynomial with $\deg(f) \geq 2$.
Let $1/2 < r < 1$. Let τ be the normalized Lebesgue measure on

$$Y_0 = \{z - \lambda \frac{f(z)}{f'(z)} \in \mathsf{Rat} \mid \lambda \in \mathbb{C}, |\lambda - 1| \leq r\} \cong \{\lambda \in \mathbb{C} \mid |\lambda - 1| \leq r\}.$$

Then τ is weakly mean stable and satisfies the assumptions of Theorems 2 and 3.
Also, for each $z_0 \in \mathbb{C} \setminus \{z \in \mathbb{C} \mid f'(z) = 0\}$,
there exists a Borel subset B_{z_0} of $(Y_0)^{\mathbb{N}}$ with $(\otimes_{n=1}^{\infty}\tau)(B_{z_0}) = 1$ satisfying the following.

- *For each $\gamma = (\gamma_1, \gamma_2, \ldots) \in B_{z_0}$,*
  there exists a $x = x(z_0, \gamma)$ with $f(x) = 0$ such that

$$\gamma_n \circ \cdots \circ \gamma_1(z_0) \to x \text{ as } n \to \infty \text{ exponentially fast.}$$

Remark 7. The statement of Theorem 6 cannot hold for deterministic Newton's method.

Idea of Proofs of Theorems 2,3.

(1) Let $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ be weakly mean stable and let $\{U_j\}_j$, $D_\tau = \bigcap_{h \in G_\tau} h^{-1}(\hat{\mathbb{C}} \setminus \cup_j U_j)$ be as in the definition of weak mean stability.

(2) In each U_j, all maps of the system are uniformly contracting and we have very nice situations in $\cup_j U_j$, e.g. There are only finitely many minimal sets of τ which meet $\cup_j U_j$ and they are "attracting".

(3) For each $y \in \hat{\mathbb{C}}$, let
$A_{y,1} :=$
$\{\gamma = (\gamma_1, \gamma_2, \ldots,) \in (\text{supp}\,\tau)^{\mathbb{N}} \mid \exists n \in \mathbb{N} \text{ s.t.} \gamma_n \circ \cdots \circ \gamma_1(y) \in \cup_j U_j\}$
and let $A_{y,2} := (\text{supp}\,\tau)^{\mathbb{N}} \setminus A_{y,1}$.

For elements in $A_{y,1}$, we have nice things (see (2)).

Regarding $A_{y,2}$, we show that for $(\otimes_{n=1}^{\infty} \tau)$-a.e. $(\gamma_1, \gamma_2, \ldots,) \in A_{y,2}$, we have $d(\gamma_n \circ \cdots \circ \gamma_1(y), D_\tau) \to 0$ as $n \to \infty$.

Idea of Proofs of Theorems 5,6.

(1) We use complex analysis, Montel's theorem (a family of uniformly bounded holomorphic functions on a domain is equicontinuous on the domain), hyperbolic metric.

(2) We classify minimal sets and analyse the bifurcation of minimal sets. etc. By using these, enlarging the support of the original τ a little bit, we destroy non-attracting minimal sets which do not meet D_τ.

(3) Regarding the proof Theorem 6, by using some technical argument, we destroy any minimal set which contains an attracting periodic cycle of $N_f(z) = z - f(z)/f'(z)$ with period ≥ 2.

Summary

(1) We introduce the notion of weak mean stability in i.i.d. random (holomorphic) 1-dimensional dynamical systems.

(2) If a random holomorphic dynamical system on $\hat{\mathbb{C}}$ is weakly mean stable, then for any $x \in \hat{\mathbb{C}}$, the orbit of the Dirac measure at x under the iterations of the dual map of the transition operator converges to a periodic cycle of probability measures.

(3) If a random holomorphic dynamical system on $\hat{\mathbb{C}}$ is weakly mean stable and satisfies some mild assumtions, then for all but countably many $z \in \hat{\mathbb{C}}$, for a.e. orbit starting with z, the Lyapunov exponent is negative. Note that the statements of (2) and (3) cannot hold for deterministic dynamics of a single rational map f with $\deg(f) \geq 2$.

(4) In many holomorphic families of rational maps (including random relaxed Newton's methods family), generic random dynamical systems satisfy the statements of (2) and (3). We can apply this to random relaxed Newton's method to find a root of any polynomial.

References:

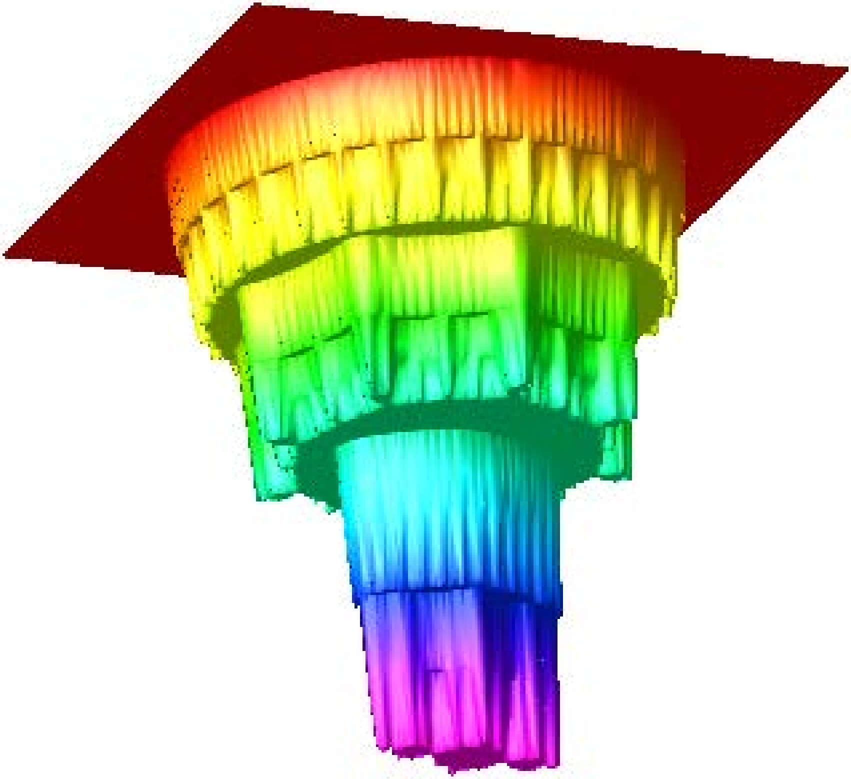
[1] R. Mañé, *The Hausdorff dimension of invariant probabilities of rational maps*, Dynamical Systems (Valparaiso, 1986) (Lecture Notes in Mathematics vol 1331) (Berlin: Springer) pp 86-117, 1988.

[2] H. Sumi, *Random complex dynamics and semigroups of holomorphic maps*, Proc. London Math. Soc. (2011) 102(1), pp 50–112.

[3] H. Sumi, *Cooperation principle, stability and bifurcation in random complex dynamics*, Adv. Math., 245 (2013) pp 137–181.

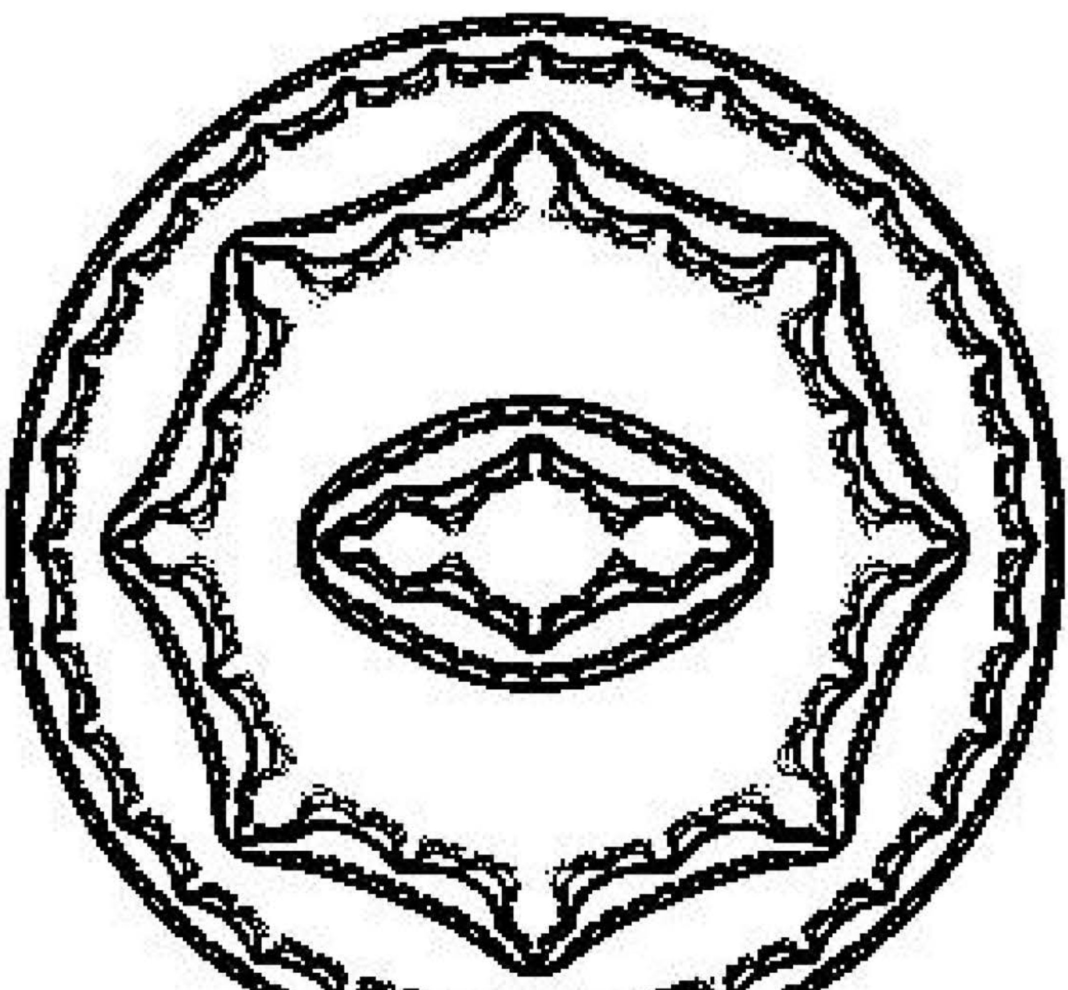
[4] H. Sumi, *Negativity of Lyapunov Exponents and Convergence of Generic Random Polynomial Dynamical Systems and Random Relaxed Newton's Methods*, 61 pages, https://arxiv.org/abs/1608.05230.
The contents of this talk are included in this paper.

Fig. 1 (A devil's coliseum)



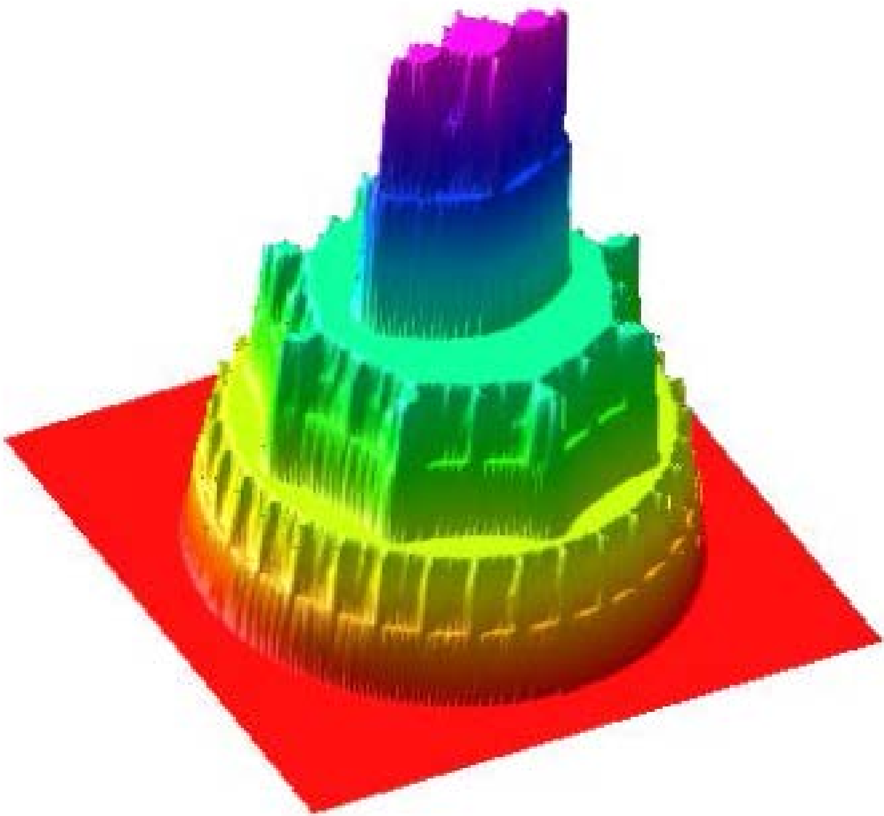
This is the graph of the function of the probability of tending to infinity regarding the randon dynamical system such that at every step we choose one of polynomials f,g of degree 4 with probabilites 1/2 and 1/2. This function is continuous on the Riemann sphere and varies precisely on a thin fractal set (Julia set of the semigroup generated by f,g) in Fig. 2. A devil's coliseum (a complex analogue of the devil's staircase).

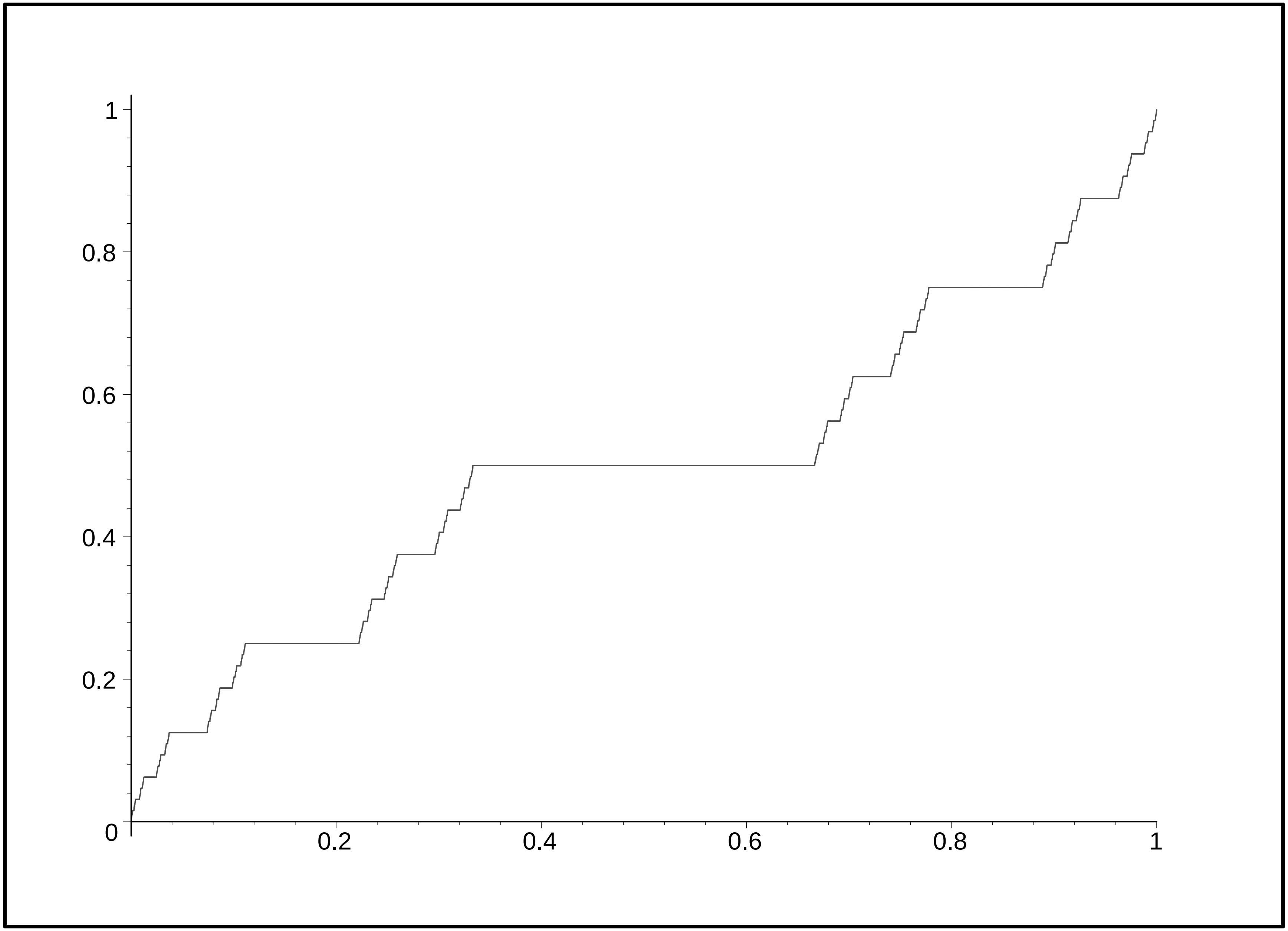
Fig. 2



The Julia set of the semigroup generated by the polynomials f and g in Fig.1. The Hausdorff dimension of the Julia set is strictily less than 2.
In particular, the 2-dim Lebesgue measure is equal to zero.
The function in Fig. 1 is continuous on the Riemann sphere and veries precisely on this Julia set.

Fig. 3. Upside down figure of Fig. 1. This is equal to the graph of the function of probability of tending to the origin. This function is continuous on the Riemann sphere and varies precisely on the thin fractal set in Fig. 2. A `` fractal wedding cake'' ([2]).





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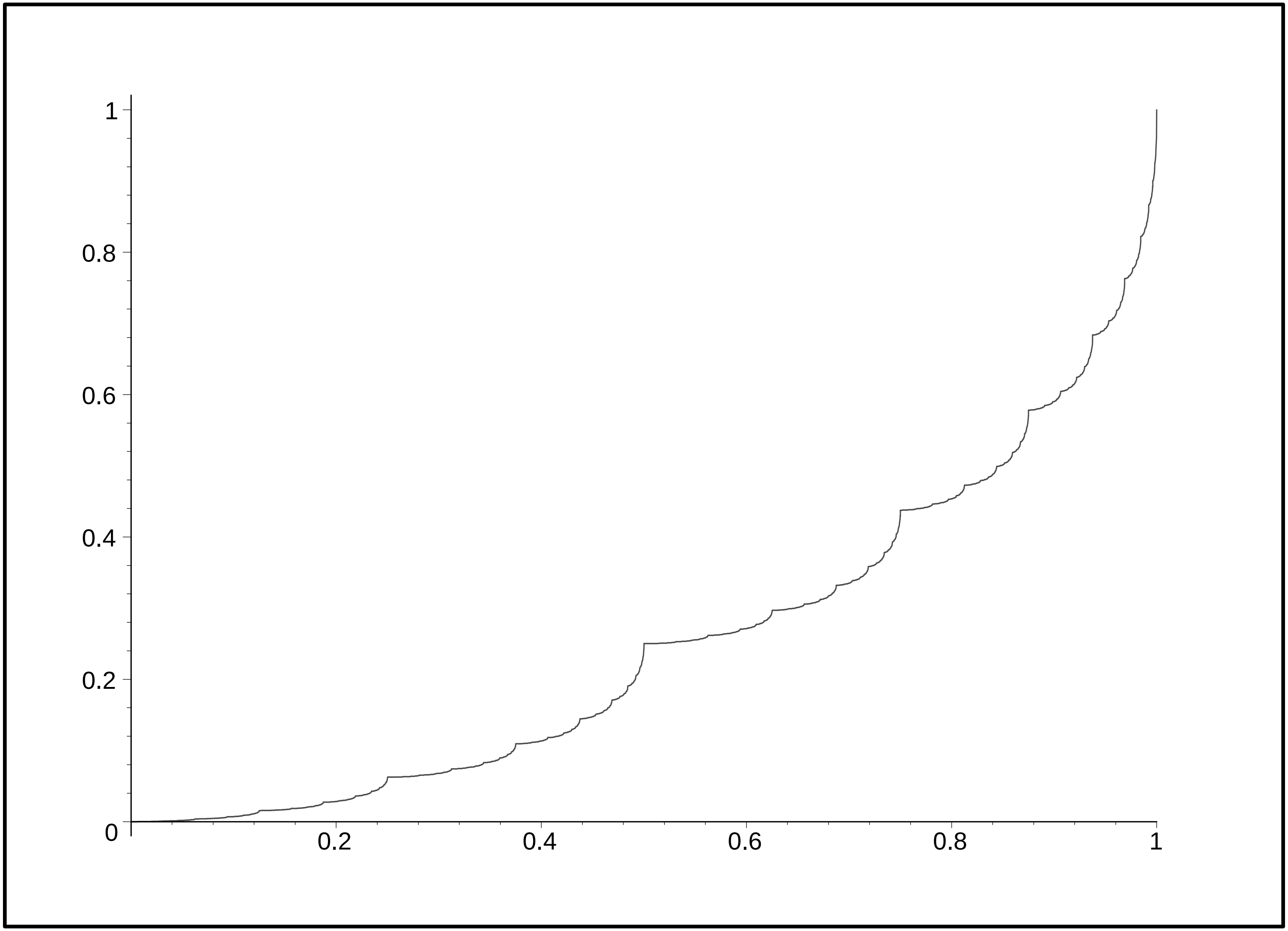

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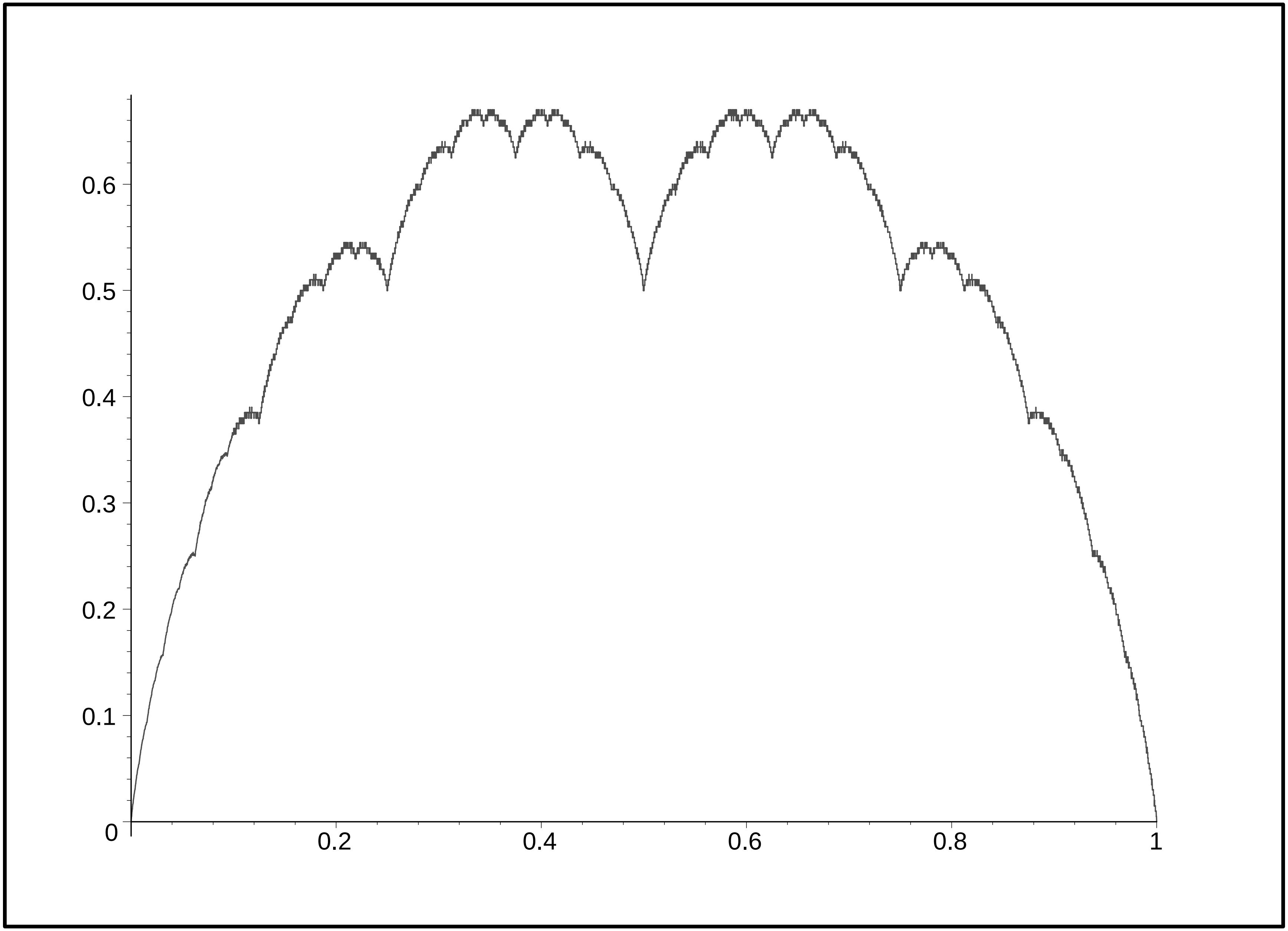
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