Böttcher coordinates for holomorphic skew products

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§.1 **Introduction**

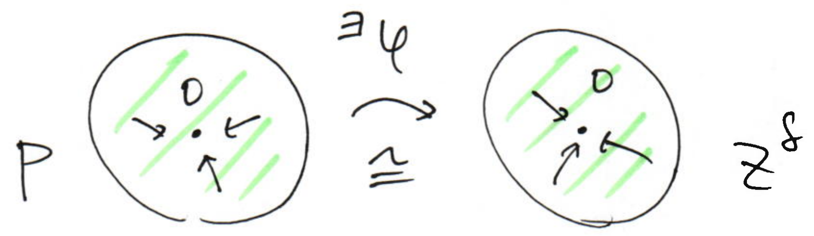
1-dim

Let p be a holo germ with a super-attracting fixed point at the origin,

$$p(z) = z^{\delta} + O(z^{\delta+1}),$$

where $\delta \geq 2$, and let $p_0(z) = z^{\delta}$.

Theorem 1 (Böttcher, 1904). .
There is a conformal germ φ defined near the origin, that conjugates p to p_0.



In fact, $\varphi(z) = \lim_{n\to\infty} \sqrt[\delta^n]{p^n(z)}$. In other words,

$$\boxed{\varphi = \lim_{n\to\infty} p_0^{-n} \circ p^n}$$

Böttcher coordinate for p

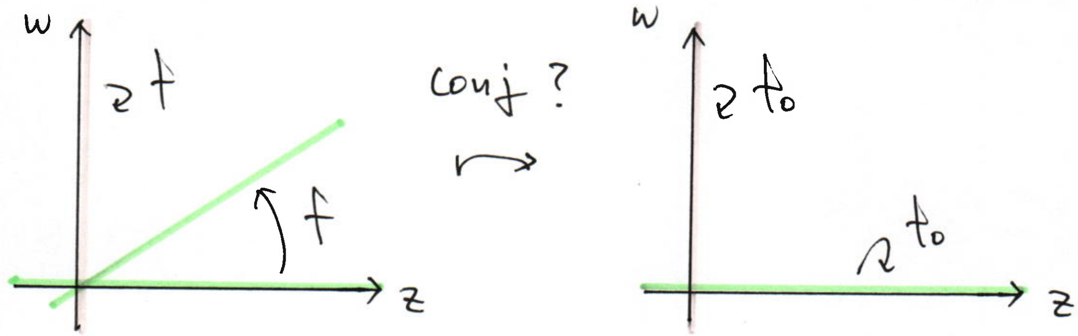
2-dim

Can we generalize Böttcher's theorem in dim 1 to dim 2? (Is a super-attr holo germ conjugate to a normal form on a neighborhood?)

Yes for some specific germs or maps
[Ushiki, 1992], [Ueda, 1993], [Favre, 2000], [Buff, Epstein and Koch, 2012]

No in general ([Hubbard and Papadopol, 1994])

Let $f(z,w) = (z^2, w^2 + cz^2)$ and $f_0(z,w) = (z^2, w^2)$. Is f conjugates to f_0 on a nbd of the origin?



$C_f = \{z=0\} \cup \{w=0\}$, $\{z=0\}$ invariant under f, $\{w=0\} \to \{w = cz\}$ under f

$C_{f_0} = \{z=0\} \cup \{w=0\}$, each invariant under f_0

The orbit of the crit set is a obstruction.

Rigid germs ([Favre, 2000])

Favre classified 2-dim attracting rigid germs.

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a holo germ,
C_f the crit set of f, and $C_f^\infty = \cup_{n \geq 0} f^{-n}(C_f)$.

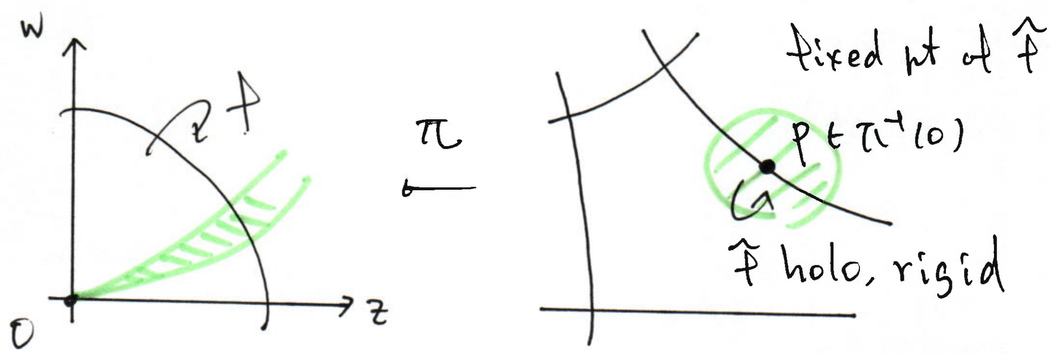
We say that f is rigid if

(i) $C_f^\infty \subset \{zw = 0\}$ in some coord, and

(ii) C_f^∞ is forward f-invariant.

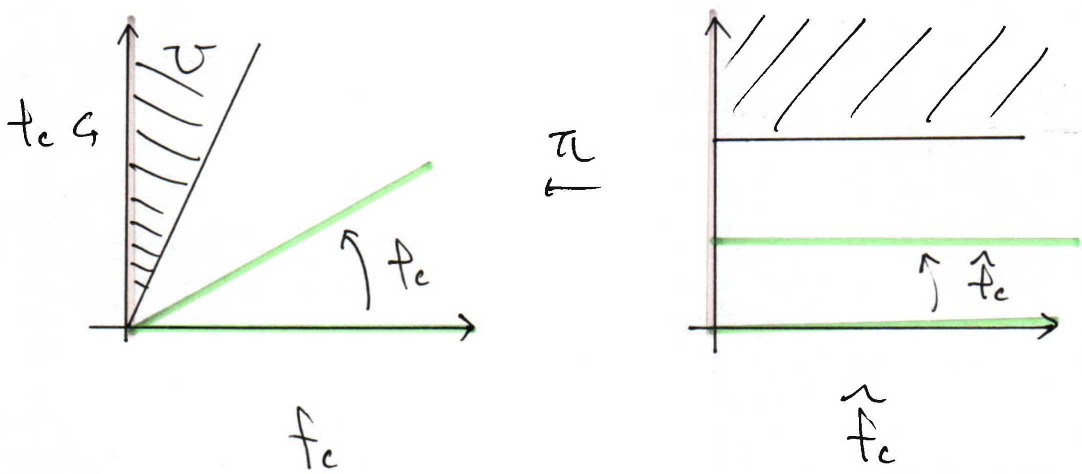
Rigidification ([Favre and Jonsson, 2007])

Any 2-dim super-attr holo germ f can be blown-up to a rigid holo germ with a fixed point at infinity.
Therefore, f is conjugate to a normal form on an open set whose closure contains the super-attr fixed pt.



Example

Let $f_c(z,w) = (z^2, w^2 + cz^2)$ and $f_0(z,w) = (z^2, w^2)$. Then f_c is conjugate to f_0 on an open set $U = \{|z| < r|w|, |w| < r\}$ for small $r > 0$, because f_c is semiconjugate to the poly product $\tilde{f}_c(z,w) = (z^2, w^2 + c)$ by $\pi(z,w) = (z, zw)$.



The dyn of f_c is determined by the dyn of $w \to w^2 + c$, which depends on the parameter c.

Theorem ([Favre and Jonsson, 2007])
Any 2-dim super-attr holo germ f is conjugate to a normal form on an open set whose closure contains the super-attr fixed pt.

Theorem
Any holo skew product with a super-attracting fixed point is conjugate to a monomial map on an invariant open set whose closure contains the super-attr fixed pt.

§.2 **Results**

Skew product:

$$f(z,w) = (p(z), q(z,w))$$

Let f be a holo skew product with a super-attracting fixed point at the origin ($f(0) = (0)$ and the eigenvalues of $Df(0)$ are both 0):

$$p(z) = z^\delta + O(z^{\delta+1}), \; \delta \geq 2$$

$$q(z,w) = Cz + \sum_{i,j \geq 0, \; i+j \geq 2} C_{ij} z^i w^j$$

Let us denote $q(z,w) = \sum C_{ij} z^i w^j$ for short.

Question: $f \sim (z^\delta, ??)$ on ??

The dominant term of p is z^δ. There is a "dominant" term $C_{\gamma d} z^\gamma w^d$ of q determined by the degree δ of p and the Newton polygon of q.

We may assume $p(z) = z^\delta$:

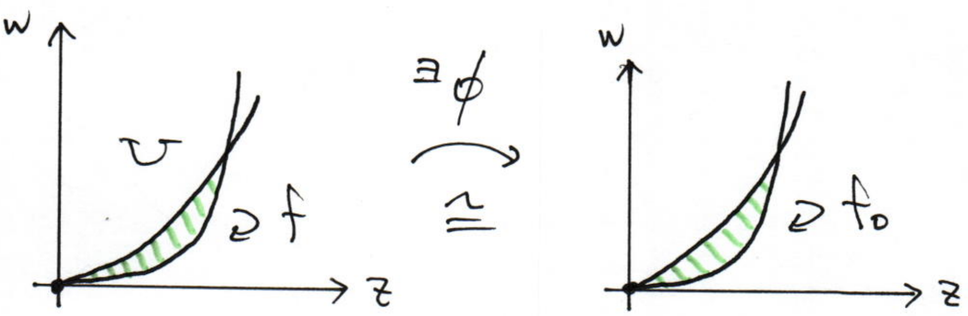
$$f(z,w) = (z^\delta, C_{\gamma d} z^\gamma w^d + \sum_{(i,j)\neq(\gamma,d)} C_{ij} z^i w^j)$$

Let $f_0(z,w) = (z^\delta, C_{\gamma d} z^\gamma w^d)$.

Theorem 2. *If $d \geq 2$, or if $d = 1$ and $\delta \neq T_k$ for any k, then $\exists$ biholo map ϕ defined on U that conjugates f to f_0, where*

$$U = \{|z|^{l_1+l_2} < r^{l_2}|w|, |w| < r|z|^{l_1}\}$$

for some $0 \leq l_1 < \infty$, $0 < l_2 \leq \infty$, and $r > 0$.



We can construct ϕ as the same as 1-dim:

$$\boxed{\phi = \lim_{n\to\infty} f_0^{-n} \circ f^n}$$

Böttcher coordinate for f on U

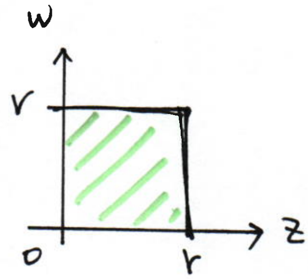
Shape of $U = \{|z|^{l_1+l_2} < r^{l_2}|w|, |w| < r|z|^{l_1}\}$

Note that if $l_2 = \infty$, then $l_2^{-1} = 0$ and so $|z|^{l_1+l_2} < r^{l_2}|w|$

$$\Rightarrow |z|^{l_1 l_2^{-1}+1} < r|w|^{l_2^{-1}} \Rightarrow |z|^{0+1} < r|w|^{0} \Rightarrow |z| < r.$$

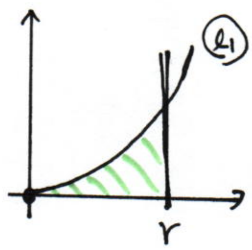
(i) If $l_1 = 0 = l_2^{-1}$, then

$U = \{|z| < r, |w| < r\}$.



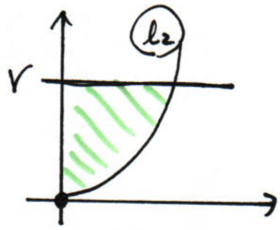
(ii) If $l_1 \neq 0 = l_2^{-1}$, then

$U = \{|z| < r, |w| < r|z|^{l_1}\}$.



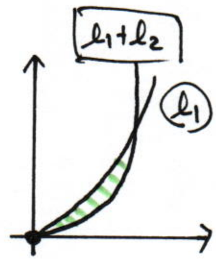
(iii) If $l_1 = 0 \neq l_2^{-1}$, then

$U = \{|z|^{l_2} < r^{l_2}|w|, |w| < r\}$.



(iv) If $l_1 \neq 0 \neq l_2^{-1}$, then

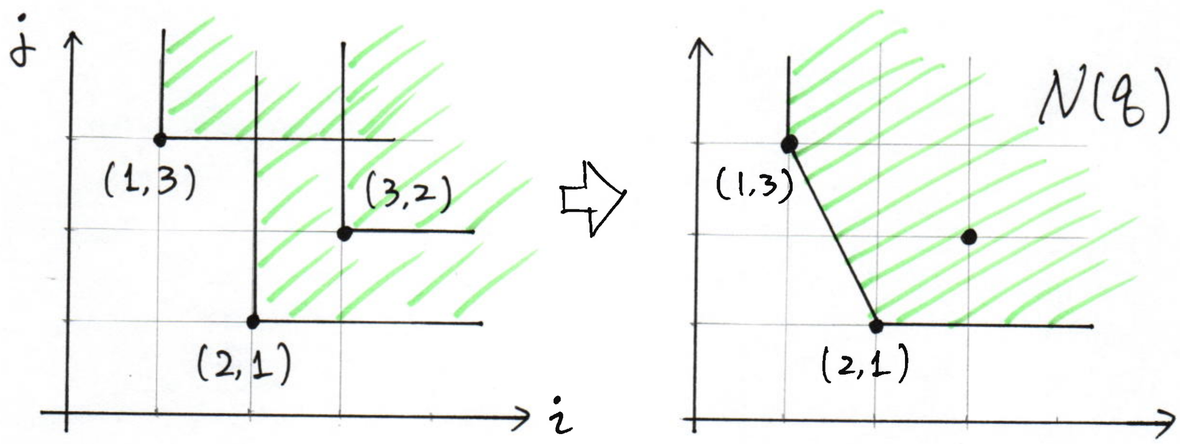
$U = \{r^{-l_2}|z|^{l_1+l_2} < |w| < r|z|^{l_1}\}$.



Newton polygon $N(q)$ of q

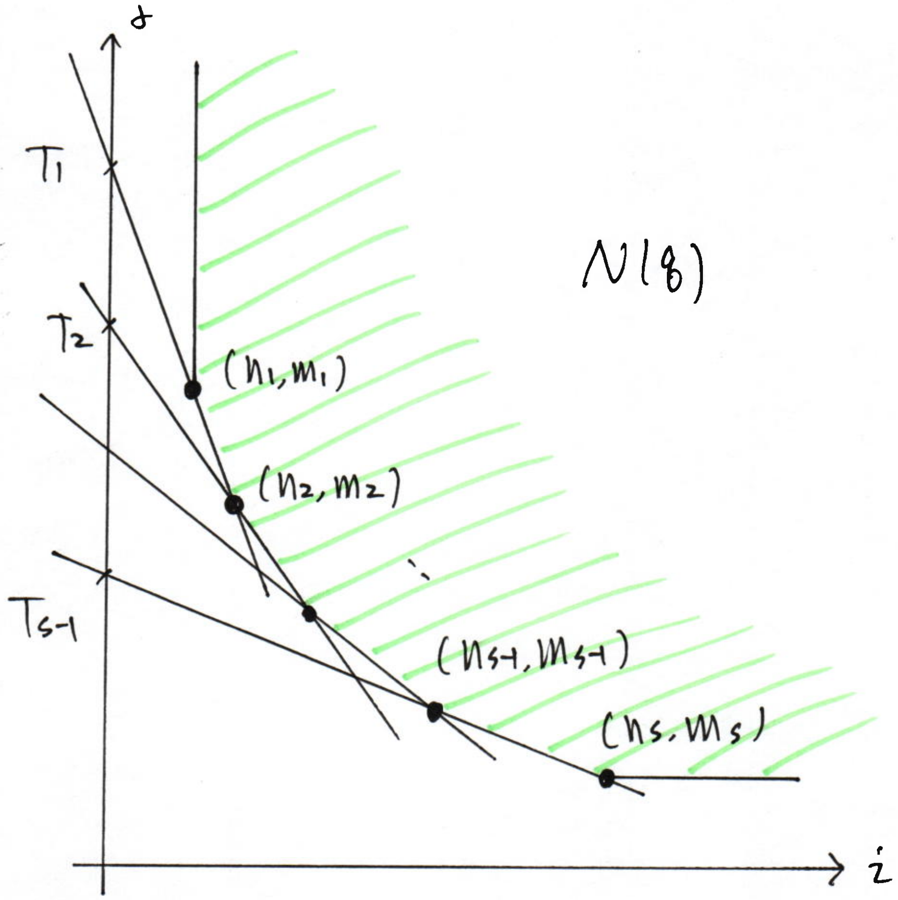
Let $q(z,w) = \sum C_{ij} z^i w^j$. We define $N(q)$ as the convex hull of the union of $D(i,j)$ with $C_{ij} \neq 0$, where $D(i,j) = \{(x,y) : x \geq i, y \geq j\}$.

If $q(z,w) = z^2 w + 10zw^3 - 5z^3 w^2$, then $N(q)$ is ...

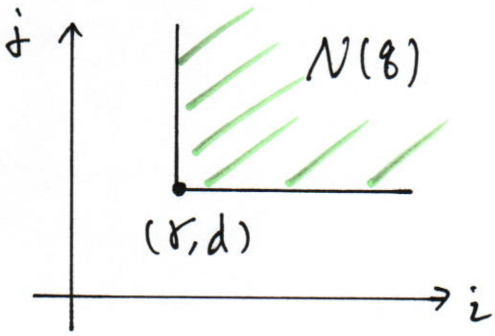


Let (n_1, m_1), (n_2, m_2), $\cdots, (n_s, m_s)$ be the vertices of the Newton polygon of q, where $n_1 < n_2 < \cdots < n_s$ and $m_1 > m_2 > \cdots > m_s$.

Let T_k be the y-intercept of the line L_k passing the vertices (n_k, m_k) and (n_{k+1}, m_{k+1}) for each $1 \le k \le s-1$.



Case (i) If $s = 1$, then $N(q) = D(\gamma, d)$.



⇨ f is conj to f_0 on a nbd.

If $s = 1$ and $d \geq 2$, then

$$f(z, w) = (z^{\delta}(1 + o(1)), z^{\gamma} w^{d}(1 + o(1)))$$

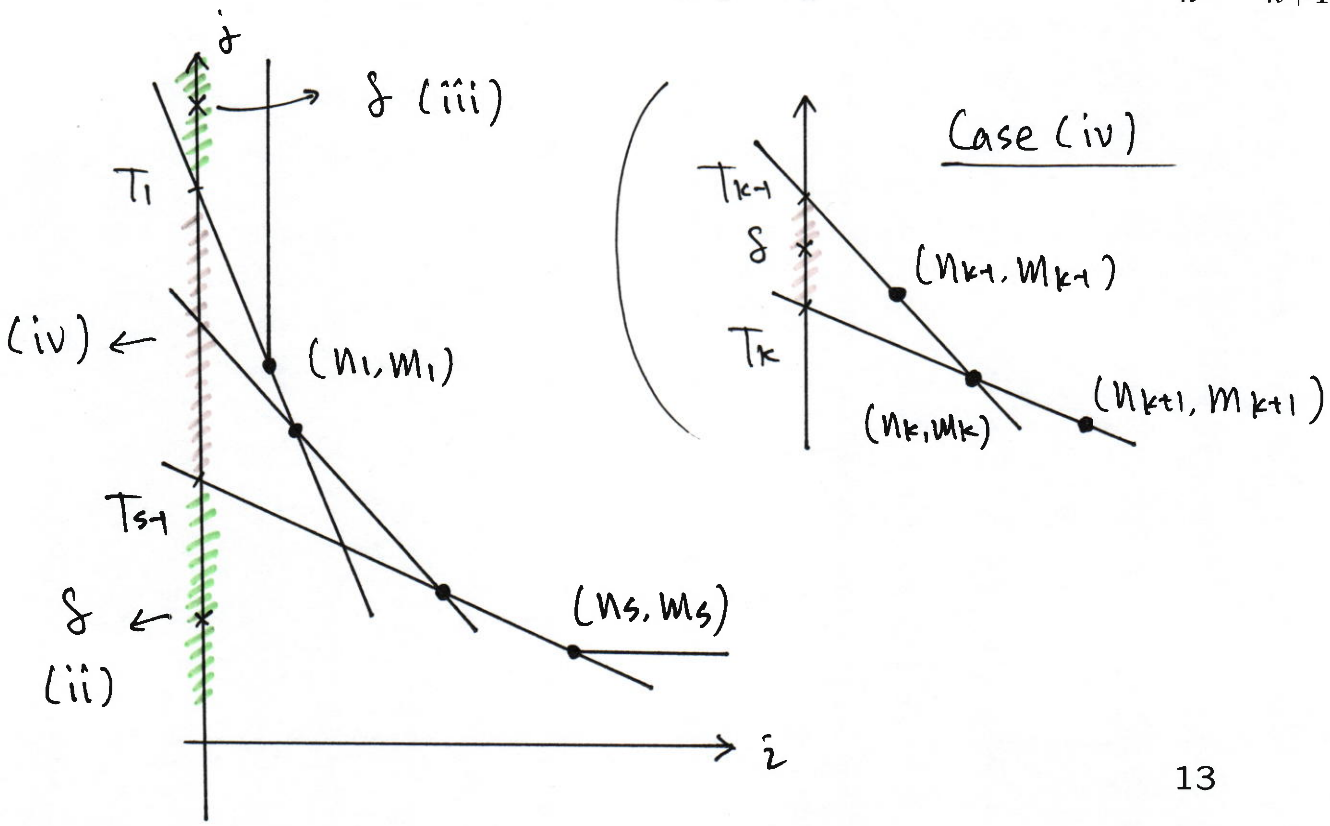
and the result is classical.

Assume that $s > 1$.

Case (ii) If $\delta \leq T_{s-1}$, then
$(\gamma, d) = (n_s, m_s)$, $l_1 = \frac{n_s - n_{s-1}}{m_{s-1} - m_s}$ and $l_2^{-1} = 0$.

Case (iii) If $T_1 \leq \delta$, then
$(\gamma, d) = (n_1, m_1)$, $l_1 = 0$ and $l_2 = \frac{n_2 - n_1}{m_1 - m_2}$.

Case (iv) If $T_k \leq \delta \leq T_{k-1}$ for $2 \leq k \leq s-1$, then
$(\gamma, d) = (n_k, m_k)$, $l_1 = \frac{n_k - n_{k-1}}{m_{k-1} - m_k}$ and $l_1 + l_2 = \frac{n_{k+1} - n_k}{m_k - m_{k+1}}$.



§.3 **Idea of Proof: blow-ups**

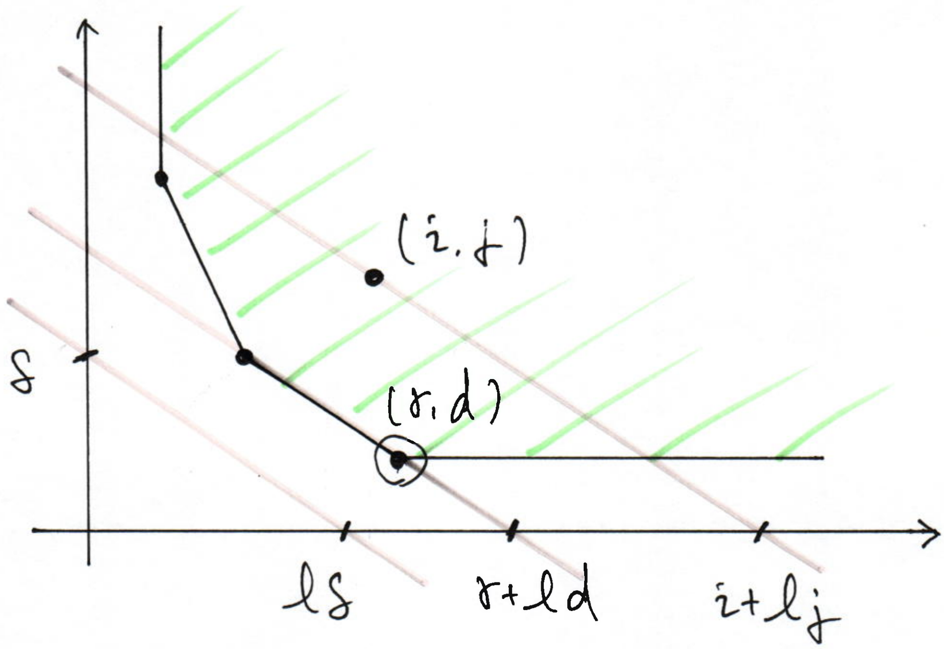
Let $d \geq 2$ and $C_{\gamma d} = 1$.
Assume that $l_1 \in \mathbb{N}$ and $l_2^{-1} \in \mathbb{N}$.

Case (ii) Let $\delta \leq T_{s-1}$, $(\gamma, d) = (n_s, m_s)$ and $l = l_1 = \frac{n_s - n_{s-1}}{m_{s-1} - m_s}$.

Lemma 1. *For any* (i, j) *such that* $C_{ij} \neq 0$,

$$l\delta \leq \gamma + ld \leq i + lj$$

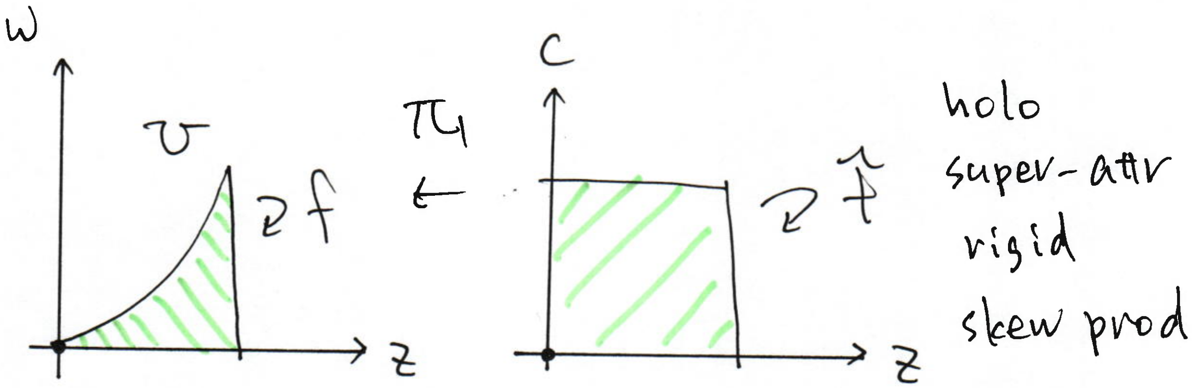
Proof. These numbers $l\delta$, $\gamma + ld$ and $i + lj$ are the x-intercepts of the lines of sloop l passing the points $(0, \delta)$, (γ, d) and (i, j). □



By Lemma 1, $l\delta \leq \gamma + ld \leq i + lj$.

Blow-up (Case (ii))

Let $\pi_1(z,c) = (z, z^l c)$ and $\tilde{f} = \pi_1^{-1} \circ f \circ \pi_1$. Then



$$\tilde{f}(z,c) = \left(p(z),\ \frac{q(z, z^l c)}{p(z)^l} \right)$$

$$= \left(z^\delta,\ z^{\gamma + ld - l\delta} c^d \left\{ 1 + \sum C_{ij} z^{(i+lj)-(\gamma+ld)} c^{j-d} \right\} \right)$$

$$= \left(z^\delta,\ z^{\gamma + ld - l\delta} c^d \{1 + o(1)\} \right).$$

Because $\tilde{f}$ is in case (i),

$\exists \tilde{\phi}$ for $\tilde{f}$ on a nbd $\Rightarrow$ $\exists \phi$ for f on U.

Newton polygon (Case (ii))

Recall that $q(z,w) = z^{\gamma}w^{d} + \sum C_{ij}z^{i}w^{j}$ and

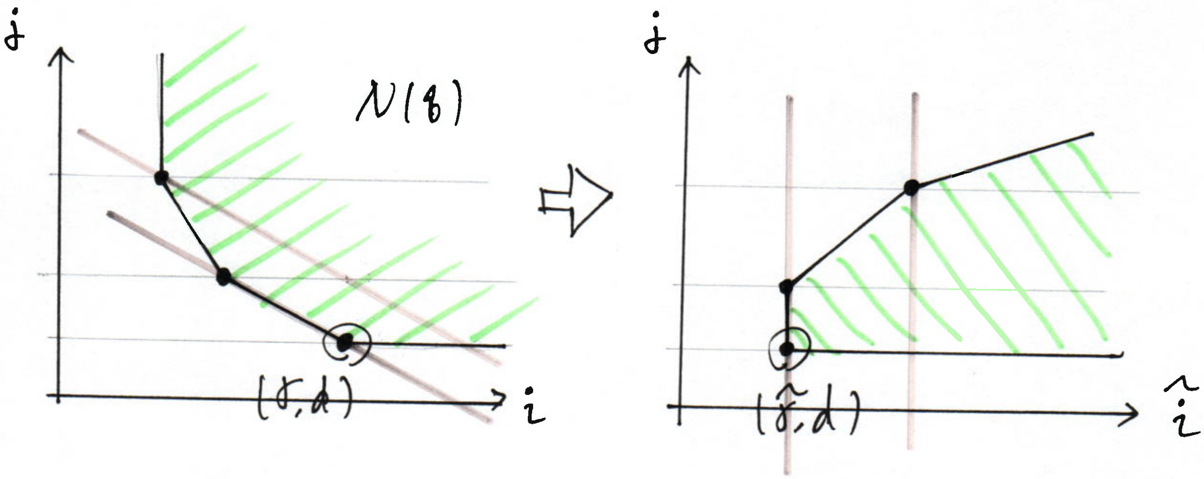
$$q(z,z^{l}c)/p(z)^{l} = z^{\gamma+ld-l\delta}c^{d} + \sum C_{ij}z^{i+lj-l\delta}c^{j}.$$

Let $\tilde{\gamma} = \gamma + ld - l\delta$ and $\tilde{i} = i + lj - l\delta$. Then

$$\begin{pmatrix}\tilde{i}\\ j\end{pmatrix} = \begin{pmatrix}i+lj-l\delta\\ j\end{pmatrix} = \begin{pmatrix}1 & l\\ 0 & 1\end{pmatrix}\begin{pmatrix}i\\ j\end{pmatrix} - \begin{pmatrix}l\delta\\ 0\end{pmatrix},$$

$$\tilde{q}(z,c) := q(z,z^{l}c)/p(z)^{l} = z^{\tilde{\gamma}}c^{d} + \sum C_{ij}z^{\tilde{i}}c^{j},$$

$$0 \le \tilde{\gamma} \le \tilde{i},\ d \le j \text{ and } N(\tilde{q}) = D(\tilde{\gamma}, d).$$



Remark (Case (ii))

Even if l_1 is a rational number, a similar statement holds (and we obtain the result).

Let $\pi_1(z,c) = (z^r, z^s c)$ and $\tilde{f} = \pi_1^{-1} \circ f \circ \pi_1$, where $s/r = l_1$. Then

$$\tilde{f}(z,c) = \left(p(z^r)^{1/r}, \ \frac{q(z^r, z^s c)}{p(z)^{s/r}} \right)$$

$$= \left(z^\delta, \ z^{r\gamma+sd-s\delta} c^d \left\{ 1 + \sum C_{ij} z^{(ri+sj)-(r\gamma+sd)} c^{j-d} \right\} \right)$$

$$= \left(z^\delta, \ z^{r\gamma+sd-s\delta} c^d \{1 + o(1)\} \right).$$

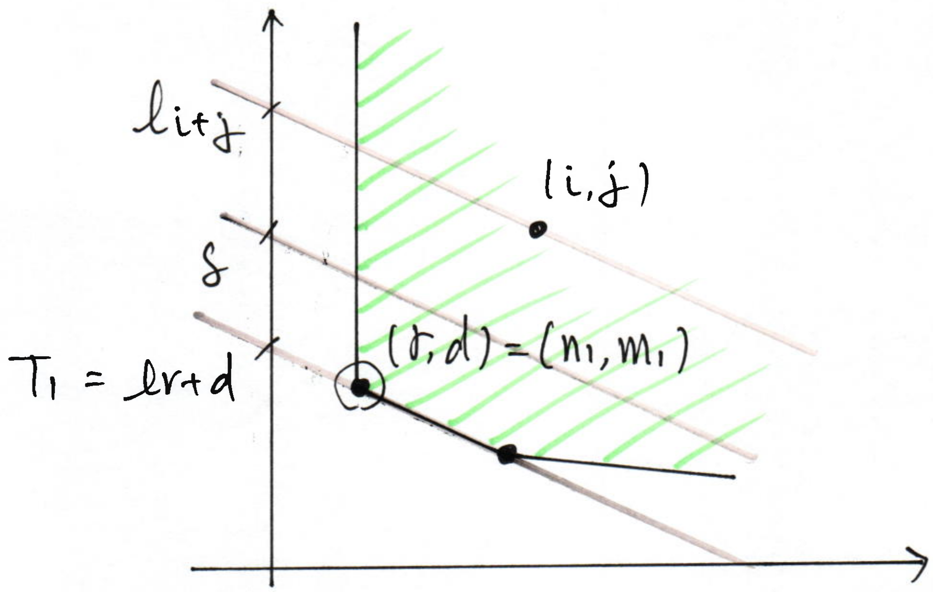
Formally, π_1 is the comp of (z^r, c) and $(z, z^{s/r} c)$.

Case (iii) Let $T_1 \leq \delta$,
$(\gamma, d) = (n_1, m_1)$ and $l = l_2^{-1} = \frac{m_1 - m_2}{n_2 - n_1}$.

Lemma 2. *For any (i, j) such that $C_{ij} \neq 0$,*

$$l\gamma + d \leq li + j \quad \text{and} \quad l\gamma + d \leq \delta$$

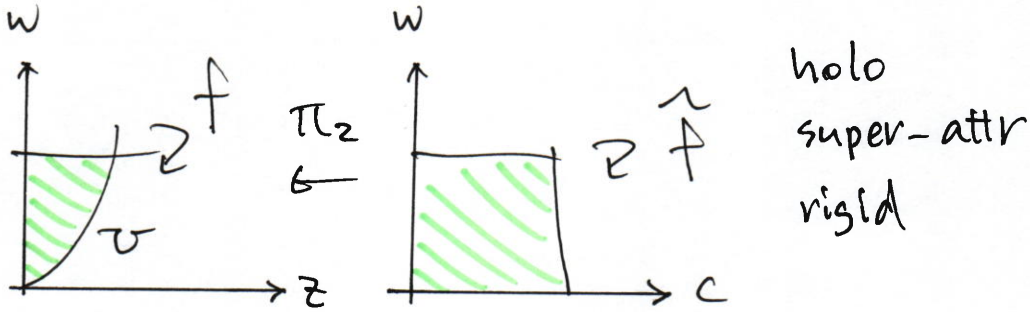
Proof. These numbers δ, $l\gamma + d$ and $li + j$ are the x-intercepts of the lines of sloop l passing the points $(0, \delta)$, (γ, d) and (i, j). □



By Lemma 2, $l\gamma + d \le li + j$ and $l\gamma + d \le \delta$.

Blow-up (Case (iii))

Let $\pi_2(c, w) = (cw^l, w)$ and $\tilde{f} = \pi_2^{-1} \circ f \circ \pi_2$. Then



$$q(cw^l, w) = c^{\gamma} w^{l\gamma+d} \left\{1 + \sum C_{ij} c^{i-\gamma} w^{(li+j)-(l\gamma+d)}\right\}$$

$$= c^{\gamma} w^{l\gamma+d}\{1 + o(1)\} \sim c^{\gamma} w^{l\gamma+d} \text{ and so}$$

$$\tilde{f}(c, w) = \left(\frac{p(cw^l)}{q(cw^l, w)^l},\ q(cw^l, w)\right)$$

$$\sim (c^{\delta - l\gamma} w^{l\{\delta-(l\gamma+d)\}},\ c^{\gamma} w^{l\gamma+d}).$$

Because of the form of $\tilde{f}$,

$$\exists\tilde{\phi} \text{ for } \tilde{f} \text{ on a nbd} \Rightarrow \exists\phi \text{ for } f \text{ on } U$$

Newton polygon (Case (iii))

Recall that $q(z,w) = z^{\gamma}w^{d} + \sum C_{ij}z^{i}w^{j}$ and

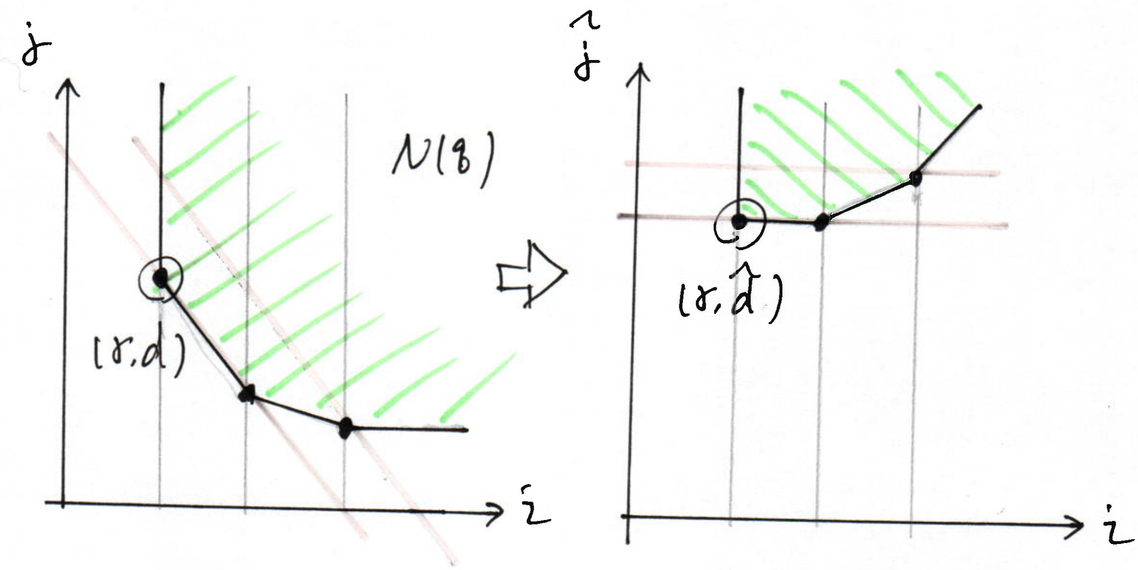
$$q(cw^{l},w) = c^{\gamma}w^{l\gamma+d} + \sum C_{ij}c^{i}w^{li+j}.$$

Let $\tilde{d} = l\gamma + d$ and $\tilde{j} = li + j$. Then

$$\begin{pmatrix} i \\ \tilde{j} \end{pmatrix} = \begin{pmatrix} i \\ li+j \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix},$$

$$\tilde{q}(c,w) := q(cw^{l},w) = c^{\gamma}w^{\tilde{d}} + \sum C_{ij}c^{i}w^{\tilde{j}},$$

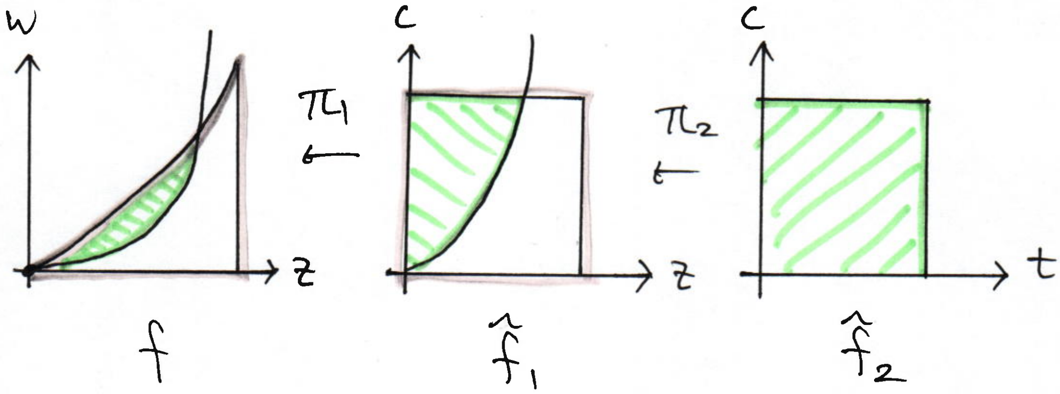
$\gamma \le i$, $d \le \tilde{d} \le \tilde{j}$ and $N(\tilde{q}) = D(\gamma,\tilde{d})$.



Case (iv) Let $T_k \le \delta \le T_{k-1}$, $(\gamma, d) = (n_k, m_k)$, $l_1 = \frac{n_k - n_{k-1}}{m_{k-1} - m_k}$ and $l_1 + l_2 = \frac{n_{k+1} - n_k}{m_k - m_{k+1}}$.

Combining cases (ii) and (iii), we can solve case (iv).

Let $\pi_1(z, c) = (z, z^{l_1} c)$ and $\pi_2(t, c) = (tc^{l_2^{-1}}, c)$.



- $\tilde{f}_1$ is holo, super-attr, skew on a nbd. Moreover, it is in case (iii).
- $\tilde{f}_2$ is holo, super-attr, rigid on a nbd.

$$\exists \tilde{\phi}_2 \text{ on a nbd } \Rightarrow \exists \tilde{\phi}_1 \Rightarrow \exists \phi \text{ on } U$$

Newton polygon (Case (iv))

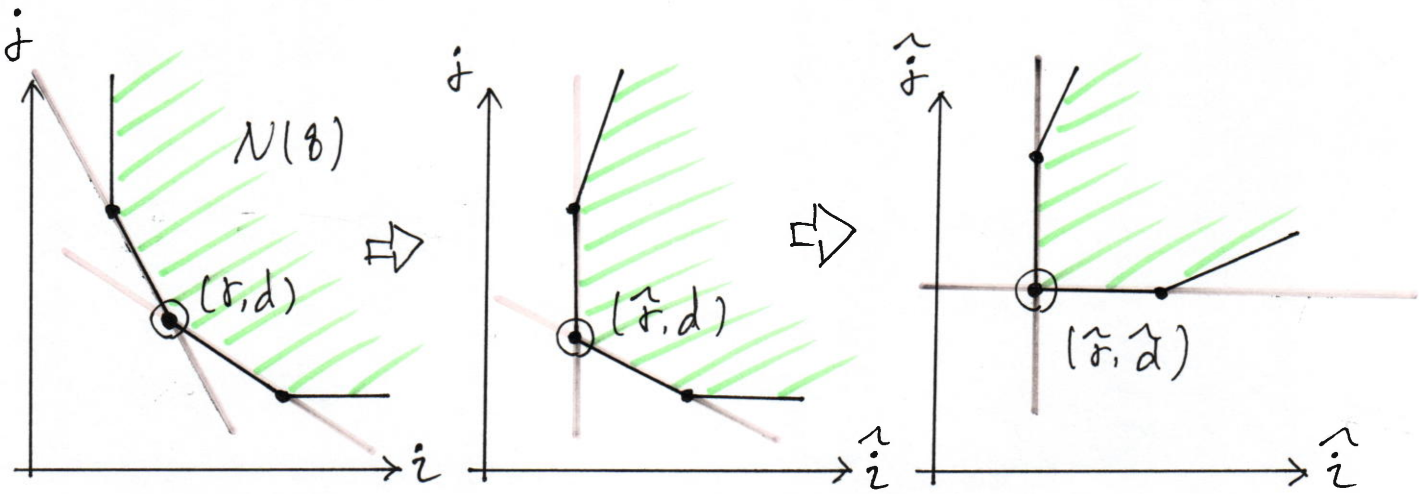
★ $q(z,w) = z^{\gamma} w^{d} + \sum C_{ij} z^{i} w^{j}$

★ $\tilde{q}_1(z,c) := q(z, z^{l_1} c)/p(z)^{l_1} = z^{\tilde{\gamma}} c^{d} + \sum C_{ij} z^{\tilde{i}} c^{j}$

$$\begin{pmatrix} \tilde{i} \\ j \end{pmatrix} = \begin{pmatrix} i + l_1 j - l_1 \delta \\ j \end{pmatrix} = \begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} - \begin{pmatrix} l_1 \delta \\ 0 \end{pmatrix}$$

★ $\tilde{q}_2(t,c) := q(t c^{l_2^{-1}}, c) = t^{\tilde{\gamma}} c^{\tilde{d}} + \sum C_{ij} t^{\tilde{i}} c^{\tilde{j}}$

$$\begin{pmatrix} \tilde{i} \\ \tilde{j} \end{pmatrix} = \begin{pmatrix} \tilde{i} \\ l_2^{-1} \tilde{i} + j \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l_2^{-1} & 1 \end{pmatrix} \begin{pmatrix} \tilde{i} \\ j \end{pmatrix}$$

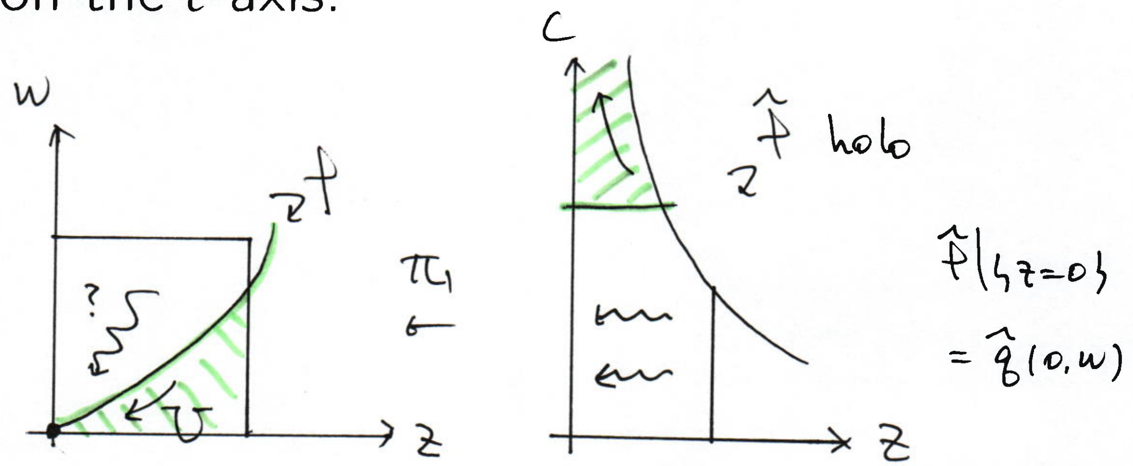


Hence $\tilde{\gamma} \le \tilde{i}$, $\tilde{d} \le \tilde{j}$ and so $N(\tilde{q}_2) = D(\tilde{\gamma}, \tilde{d})$.

§.4 **Dyn on the complement**

Dyn of f on $U \leftarrow$ Dyn of f_0
Dyn of f on a nbd$\setminus \cup f^{-n}(U) \leftarrow$??

For case (iii), we use π_1 instead of π_2. Then $\tilde{f}$ is well-def, and dyn should be well understandable; dyn of $\tilde{f}$ should be dominated by dyn of $\tilde{f}$ on the c-axis.



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