Variations of Hausdorff dimension of quadratic Julia sets

Ludwik Jaksztas, Michel Zinsmeister

Kyoto, december 15, 2017

Introduction

- We consider the quadratic family

$$f_c(z) = z^2 + c$$

- We define as usual

$$K_c = \{z \in \mathbb{C};\ (f_c^n(z))_{n\in\mathbb{N}} \textit{ is bounded}\},$$

$$J_c = \partial K_c.$$

- We define $d(c) =$ the Hausdorff dimension of J_c.
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Known results

- (Ruelle): d is real-analytic in each hyperbolic component.
- (Bodart, Z.) d is continuous from the left at $1/4$
- (Douady, Sentenac, Z.) d is discontinuous from the right at $1/4$.
- (McMullen) d is continuous at any real parabolic point with two petals (but discontinuous along the vertical direction).
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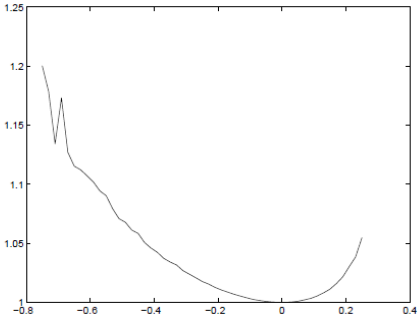

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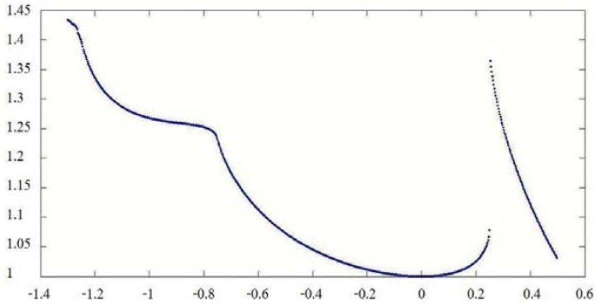

Derivative of d

- (Havard-Zinsmeister) With the information that $d(1/4) > 1$ and $d(1/4) < 3/2$, we get, for c close to $1/4$, $c < 1/4$

$$\frac{1}{K} \leq \frac{d'(c)}{(1/4 - c)^{d(1/4) - 3/2}} \leq K$$

for some $K > 1$.

- (Jaksztas) With the information that $d(-3/4) < 4/3$ we get for c close to $-3/4$

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Theorem (Jaksztas, Z.): Let c_0 be a real parameter which is parabolic with two petals: then

1. if $d(c_0) < 4/3$,

$$\lim_{c \to c_0 \pm 0} \frac{d'(c)}{|c - c_0|^{\frac{3}{2}d(c_0) - 2}} = -K_{\pm},$$

   with $K_+ > K_- > 0$.

2. if $d(c_0) = 4/3$ then

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Thermodynamic Formalism

- Let (F_λ), $\lambda \in \Lambda$ a disk $\subset \mathbb{C}$ be a holomorphic family of hyperbolic polynomials with same degree d.
- If $\lambda_0 \in \Lambda$ there exists a holomorphic motion φ_λ of $\mathcal{I}_{\lambda'}$ such that $\varphi_{\lambda_0} = id$ and

$$\varphi_\lambda \circ F_{\lambda_0} = F_\lambda \circ \varphi_\lambda.$$

- Let $X = \mathcal{I}_{\lambda'}$, $T = F_{\lambda_0}$ and consider the potential function

$$\Phi = -\log |F'_\lambda(\varphi_\lambda)|.$$

- The topological pressure is defined as

$$P(T, \Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{y \in T^{-n}(x)} \exp S_n\Phi(y),$$

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- The function $t \mapsto P(T, t\Phi)$ is convex decreasing from $+\infty$ to $-\infty$, so it vanishes once and only once.
- Theorem (Bowen): The unique zero of this function is the Hausdorff dimension of $\mathcal{I}_\lambda$.
- Ruelle operator:

$$\mathcal{L}_\Phi(u)(x) = \sum_{y \in T^{-1}(x)} \exp \Phi(y) u(y).$$

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- There exists a unique probability measure ω_Φ such that $\mathcal{L}^*(\omega_\Phi) = \beta\omega_\Phi$.
- If $\beta = 1$, i.e. if t is the dimension then

$$\mu_\Phi = h_\Phi \omega_\Phi$$

  is a T-invariant measure.
- Parry-Pollicott:

$$\frac{d}{dt}(P(T, \Psi + t\Phi)) = \frac{\int_X \Phi d\mu_{\Psi+t\Phi}}{\mu_{\Psi+t\Phi}(X)}.$$

- Let $\Phi_\lambda = -d(\lambda) \log |F'_\lambda(\varphi_\lambda)|$.
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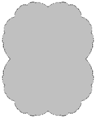
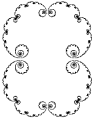
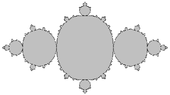
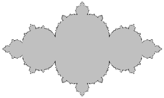




The bifurcation: Two petals case





The bifurcation: two petals case

- Let f_{c_0} have a parabolic cycle of length k. Let $\alpha(c_0)$ be an element of this cycle.
- $(f^k_{c_0})'(\alpha(c_0)) = \pm 1$ and -1 corresponds to two petals.
- Since $(f^k_{c_0})'(\alpha(c_0)) \neq 1$ there exists a nbhd U of c_0 and a holomorphic function α on U so that for $c \in U$, $\alpha(c)$ is an element of a k-cycle.
- c_0 lies between two hyperbolic components of the Mandelbrot set, W_l, W_r and we may assume that $W_l \cup W_r \subset U$. Then $\alpha(c)$ is attracting if $c \in W_r$ and repelling if $c \in W_l$.
- Now we change parameter from c to $\lambda = (f^k_c)'(\alpha(c))$. We get $F_\lambda(z) = \lambda z + az^2 + bz^3$ and we get

$$F^2_{-1}(z) = z - 2(a^2 + b)z^3 + ..$$

- Because the critical orbits of F_λ are eventually real $a^2 + b$ is positive and we call scaling factor the quantity $A = \sqrt{a^2 + b}$ (which is equal to 1 for $c_0 = -3/4$).

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"Fatou" coordinates

- Conjugating F_λ by the right Möbius map we may assume that $\pm\frac{\sqrt{-\delta_\lambda}}{A}$ are the fixed points of F_λ^2.
- We then define the "Fatou" coordinates as

$$Z_\lambda = \frac{1}{2\lambda}\log\left(1 - \frac{\delta_\lambda}{A^2 z^2}\right).$$

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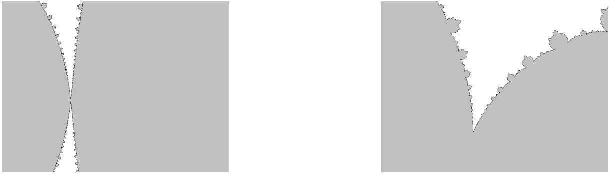

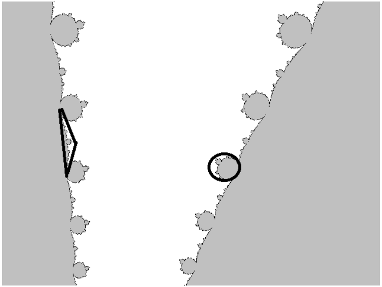

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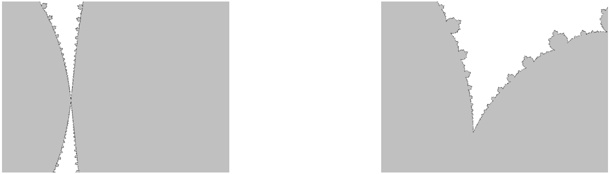
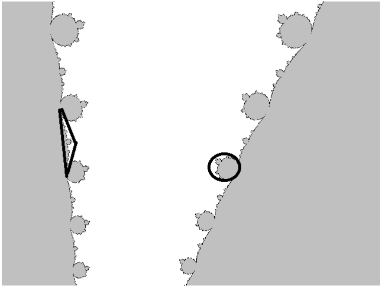
- 



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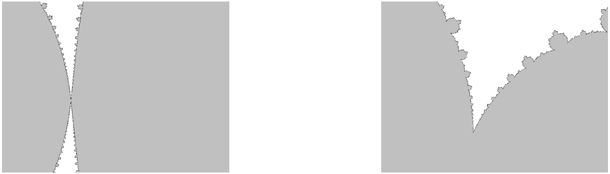
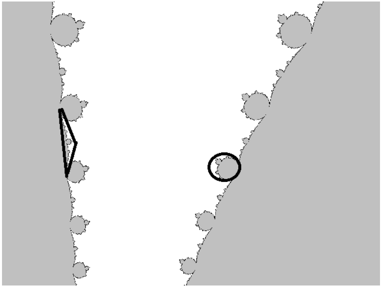
- 
- 

Cylinders

There exist $K, K_1, K_2 > 0$ and $\eta > 0$ such that for every $n \geq 1$ and $|\delta_\lambda| < \eta$ we have

$$K_1 |\delta_\lambda| e^{-Kn|\delta_\lambda|} \leq |C_n(\lambda)| \leq K_2 n^{-3/2}.$$

Invariant measure

- We consider the induced map $F^* = F^k$ on $|C_k|$.
- This induced dynamics is hyperbolic and thus there is an F^*-invariant measure μ^* which is absolutely continuous wrt the conformal (=Hausdorff) measure with Radon-Nykodim derivative bounded and away from 0.
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$$\Lambda_0^h(u) = \left(\frac{e^{-2u}}{|1 - e^{-2u}|^{3/2}} \right)^h .$$

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$$(\delta_\lambda > 0) : \frac{\mu_\lambda(C_n)}{\delta_\lambda^{3d(\lambda)/2-1}} \sim M \int_{n\delta_\lambda}^{\infty} \Lambda_0^{d(\lambda)} du,$$

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Estimate of the λ-derivative

- We return to the formula giving the derivative of the dimension: we need to estimate

$$\frac{\partial}{\partial\lambda}(F_\lambda'(\varphi_\lambda)) = (\frac{\partial}{\partial\lambda}F'\lambda)(\varphi_\lambda) + \dot{\varphi}_\lambda F_\lambda''(\varphi_\lambda).$$

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$$\dot{\varphi}_\lambda = -\sum_{j=1}^{\infty} \frac{\dot{F}_\lambda(F_\lambda^{j-1}(\varphi_\lambda))}{F_\lambda^{j}{}'(\varphi_\lambda)}.$$

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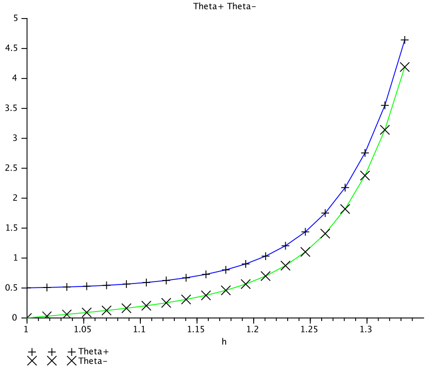

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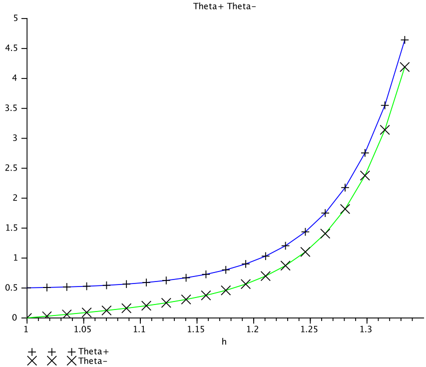
- 

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